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POSTULATES IN THE HISTORY OF SCIENCE

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J. Hadamard recently remarked that the history of science has always been and will always be one of the parts of human knowledge where progress is the most difficult to be assured.[1] This difficulty is partly due to the fact that many technical terms have been used with widely different meanings during the development of some of the scientific subjects so that statements which conveyed the accurate situation at one time sometimes fail to do so at a later time in view of the different meanings associated by later readers with some of the technical terms involved therein. Since it is obviously desirable to secure assured progress in all fields of human knowledge it seems opportune to endeavor to remove some of the difficulties which now hamper progress in the history of science. Possibly a more explicit use of postulates (a method introduced into the development of mathematics by the ancient Greeks) would also be of service here, and in the present article we shall first consider a few consequences in the history of mathematics resulting from a strict adherence to the following postulate: In a modern historical work on science the technical terms should be used only with their commonly accepted modern meanings unless the contrary is explicitly stated.

It may be instructive to consider this postulate in connection with the widely known mathematical term logarithm, which is most commonly defined in modern treatises as the exponent x of a number a, not unity, such that if y is any given number then $a^x = y$. A somewhat less commonly used definition in elementary mathematics for real logarithms is based on a (1, 1) correspondence between an arithmetic and a geometric progression such that 0 of the arithmetic progression corresponds to 1 of the geometric progression, the ratio being not equal to unity. The numbers in this arithmetic progression are then said to be the logarithms of the corresponding numbers in this geometric progression. From both of these definitions it results that the logarithm of the product of two numbers is equal to the sum of their logarithms, and this is usually regarded as a

fundamental law of logarithms. In fact, A. L. Cauchy and others studied the functional equation

$$f(xy) = f(x) + f(y)$$

and found that under certain restrictions $f(x) = \log x$ is a general solution thereof, and hence B. Riemann and others called functions which satisfy this law general logarithmic functions.

On the other hand, the numbers which J. Napier first called logarithms and which he published in the tables of his noted work entitled "Mirifici Logarithmorum Canonis Descriptio," 1614, do not obey this fundamental law of logarithms. Notwithstanding this fact mathematical historians commonly refer to these tables as tables of logarithms. In fact, they are now commonly called the earliest tables of logarithms. It is clear that the reader who has not made a special study of this question, and infers that the reasonable postulate noted above is adhered to, naturally derives from such statements an inaccurate impression as regards the actual contributions of Napier towards the development of logarithms. A statement to the effect that these tables, due to Napier, are not logarithms in the modern sense of this term, but serve some of the purposes for which logarithms are now used, and were very influential in the development thereof, would seem to give the reader a much more accurate idea of the work of Napier. The fact that Napier rendered such very effective service in spite of his serious limitations may also serve as an inspiration.

A large number of recent writers have protested againt the still somewhat common use of the term Naperian logarithms for logarithms to the base e in view of the obvious fact that the numbers of the tables which Napier called logarithms decrease as the corresponding numbers increase. According to the postulate noted above the term Naperian logarithms should now be regarded as obsolete. The fact that this term appears in the equation

$$\text{Nap. } \log x = 10^7 \log_e 10^7/x$$

which is found in many places, including Pascals's *Repertorium der höheren Mathematik*, volume 1, 1900, page 467, and also the two recent general American histories of mathematics, can scarcely be used now as an argument in its favor since the inaccuracy of this equation, according to later historical studies relating to logarithms, results directly from the striking and well-known theorem that the only pair of algebraic numbers which satisfies the equation $y = e^x$ is $x = 0, y = 1$. The tables which Napier computed imply that the rational integral numbers listed therein have rational integral logarithms, according to his use of this term. It is, therefore, obvious that no general exact rational relation can possibly exist between these so-called logarithms and those of the base e, since Napier did not consider

any operation leading to transcendental numbers. The fact that the given equation is approximately true is of secondary interest since a base which is approximately equal to e or $1/e$ can be read into any (1, 1) correspondence between a sufficiently extensive arithmetic and a geometric progression in which the ratio of the geometric progression varies only slightly from unity and the numbers of the arithmetic progression are called logarithms. Hence the fact that the bases $1/e$ and e can be read into the so-called logarithms of Napier and Bürge, respectively, does not imply any special merits thereof and is of little interest.

Although the postulate noted at the close of the first paragraph is so obvious many other difficulties in the history of mathematics could be avoided by a strict adherence thereto. We shall cite here only one more, relating to logarithms, viz., the common statement that tables to the base e were constructed by some of the immediate followers of Napier, including J. Speidell. It is well known that the number e was not determined until long after these tables were constructed, and that the limiting process involved in its determination was entirely unknown to the authors of these tables. Since a logarithmic table aims to give only a certain number of the correct decimals of the logarithms involved therein, it is obvious that an infinite number of different bases give rise to the same table. In particular, no table of common logarithms would be changed by replacing the base 10 by various other bases nearly equal to 10, and the latitude of the possible values for the base evidently increases as the number of decimal places in the table decreases. Hence, it is clearly impossible to determine the exact base of a table from the logarithms found therein, and the fact that these logarithms may agree with those constructed by the use of the base e does not now justify the statement that they are logarithms to this base, without some modification. The term logarithmic table to the base e now implies that the value of e was actually considered in its construction and hence that this value was known at the time of its construction, otherwise we could not speak of a definite base of a table of logarithms. This is a somewhat less obvious consequence of the use of the postulate in question than those noted above, and it is such sharp distinctions that make the history of science both attractive and difficult.

Among the other postulates which should perhaps be explicitly stated in a history of science is the following: A scientific concept must be clearly understood before its history can be presented in a satisfactory manner. In particular, the mathematics of the history of mathematics is the most extensive and the most difficult mathematics. Hence, historical studies relating to scientific developments naturally follow the study of the subjects themselves, and a comprehensive historical study becomes possible only after considerable maturity has been attained in the subjects concerned. This does not exclude the desirability of some efforts in this di-

rection at earlier stages. Moreover, it should be noted that while the postulation method has been very useful in development of mathematics it has not appealed equally to all mathematicians and has not proved to be a panacea here, and it can scarcely be hoped that it would prove to be one in the history of science. The few instances cited above, illustrating its use in the history of mathematics, may, however, suggest a method of overcoming some of the difficulties to which Hadamard referred. It should be added that his remarks evidently do not apply to the collections of scientific data which are now sometimes called history, since progress along this line is now reasonably rapid and easily assured. With the growth of the history of science there must go an increase of accuracy relating thereto which makes reconsiderations necessary, and reference to authorities, while often useful, may also become a menace, since they are just as formidable in the presentation of current false views as correct ones.

[1] *Bull. Sci. Math.*, **50**, 97 (1926).

ON THE ZEROS OF FUNCTIONS ASSOCIATED WITH A LINEAR SYSTEM OF THE SECOND ORDER

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The following three statements concerning the zeros of continuous functions on an interval are immediate.

If $f(x)$ is a real continuous function of x on I: $a \leqq x \leqq b$, the set of zeros of $f(x)$ on I is closed.

If $f(x)$ is not identically zero on I, the set of zeros cannot be everywhere dense on I.

If $f(x)$ is not identically zero in any subinterval of I, the set of zeros is nowhere dense on I.

We shall prove the following lemma:

If $f(x)$ has a derivative at every point of I and if $f^2 + f'^2 > 0$ on I, the set of zeros is finite in number.

Proof: If the zeros, (x_i), of $f(x)$ on I are infinite in number, the set has at least one limit point, c, which belongs to the set. Let $x_1, x_2, \ldots, x_i, \ldots$ be a set of points on I having c as its sequential limit point. Then

$$\lim_{i \to \infty} \frac{f(x_i)-f(c)}{x_i-c} = f'(c) = 0.$$

But $f(c) = f'(c) = 0$ contradicts $f^2 + f'^2 > 0$ on I. Hence there are only a finite number of zeros of $f(x)$ on I.

Concerning the linear system

$$y' = K(x)\, z(x), \quad z' = G(x)\, y(x), \tag{1}$$

$$y(a) = A, \qquad z(a) = B, \tag{2}$$

we make the following hypotheses:

1. $K(x)$ is a real continuous function of x on I, $K > 0$ on I,
2. $G(x)$ is a real bounded and integrable (Lebesgue) function of x on I,
3. A and B are real constants.

The method of successive approximations, starting with the function

$$y_0(x) = A + BK(a)\,(x-a),$$

enables us to find a unique pair of functions (y, z) with the following properties,

(*a*) (y, z) satisfy (2),

(*b*) (y, z) satisfy (1), the first equation for every x on I, the second equation almost everywhere on I (except for a set of measure zero),

(*c*) (y, z) are continuous functions of x on I,

(*d*) y' exists and is continuous on I,

(*e*) z' exists almost everywhere on I, and is bounded and integrable on I.

If $A = B = 0$, then $y \equiv 0$ and $z \equiv 0$. By a *solution* of (1) and (2) we shall mean a pair of functions (y, z) having the above properties for which $A^2 + B^2 > 0$ and hence such that $y^2 + z^2 > 0$ on I. This may be stated otherwise as follows, if $y^2 + z^2 \not\equiv 0$ on I, then $y^2 + z^2 \neq 0$ on I.

THEOREM I. *The set of zeros of a solution y of* (1) *is finite in number.* The proof follows immediately by use of the lemma above.

Let $g(x)$, $h(x)$, $g^2 + h^2 > 0$ be real functions of x on I, having bounded integrable derivatives, g', h', almost everywhere on I. Consider

$$\phi(x) = gz - hy$$

which has a bounded integrable derivative almost everywhere on I,

$$\phi'(x) = (g' - hK)z + (gG - h')y.$$

If $[gh] = g^2G - h^2K + g'h - gh'$, it follows immediately from the theory of linear homogeneous algebraic equations that a necessary and sufficient condition that $\phi^2 + \phi'^2 > 0$ on I is that $[gh] \neq 0$ on I.

THEOREM II. *If $g \neq 0$ and $[gh] > 0$ on I, then neither y nor ϕ can vanish twice on I, and if one vanishes on I, the other does not vanish on I.*

Proof: Let $\bar{\phi} = z - \bar{h}y$, where $\bar{h} = h/g$. Then the product $y\bar{\phi}$ is a con-

tinuous function with the bounded integrable derivative almost everywhere on I,

$$(y\bar{\phi})' = Kz^2 - 2K\bar{h}yz + (G - \bar{h}')y^2 = Q(y, z).$$

This quadratic form in (y, z) is positive definite, if

$$K\bar{h}^2 - G + h' < 0$$

or

$$[gh] > 0.$$

If y or $\bar{\phi}$ vanishes at x_1, we may write

$$y\bar{\phi} = \int_{x_1}^{x} Q(y, z)dt.$$

Hence $y\bar{\phi}$ is a monotone function on I, not a constant, and y and $\bar{\phi}$ cannot both vanish at x_1. Hence the theorem is proved.

If $h \equiv 0$, then take $g \equiv 1$ and $\phi \equiv z$ and we obtain the result that if $G > 0$, neither y nor z vanishes twice on I and if either vanishes once on I, the other does not vanish on I. Hence if $G > 0$, the system (1) is *non-oscillatory*.

THEOREM III. *If either* $g(x) \equiv 0$ *or* $g(x) \neq 0$ *and* $g^2G + g'h - gh' < 0$, *the set of zeros of* ϕ *on* I *is finite in number.*

Proof: If $g(x) \equiv 0$, then $h(x) \neq 0$ and the present theorem reduces to Theorem I. Otherwise, let $\bar{\phi} = z - \bar{h}y$. If the set of zeros of $\bar{\phi}$ on I is infinite in number then it has a limit point c belonging to the set. There exists a set of distinct values $c_1, c_2, \ldots, c_n, \ldots$, of the set having c as a sequential limit point, where, either $c_1 < c_2 < \ldots < c$, or $c_1 > c_2 > \ldots > c$. Consider any interval $(c_n < c)$. Now $y(c_n) \neq 0$ and $y(c) \neq 0$. If $y \neq 0$ on $c_n < x < c$, then $\bar{\phi}/y$ is continuous on (c_n, c) and has the bounded integrable derivative almost everywhere on (c_n, c),

$$(\bar{\phi}/y)' = G - \bar{h}' - Kz^2/y^2.$$

If $G - \bar{h}' < 0$, then $(\bar{\phi}/y)' < 0$ on (c_n, c). Now

$$(\bar{\phi}/y) = \int_{c_n}^{x} (G - \bar{h}' - Kz^2/y^2)dt$$

is monotonic on (c_n, c), and $(\bar{\phi}/y) = 0$ at $x = c_n$. This contradicts $\bar{\phi}(c) = 0$. Hence y must vanish in every interval $c_n < x < c$ and hence an infinite number of times on I. This contradicts theorem I and proves the present theorem, if $G - \bar{h}' < 0$.

Now $g^2(G - \bar{h}') = g^2G + g'h - gh'$ which gives the condition of theorem III. If $[gh] > 0$, ϕ has at most one zero on I. If $g^2G + g'h - gh' < 0$, it will also be true that $[gh] < 0$.

If $\bar{h} \equiv 0$, we obtain the special result that the zeros of z are finite in number on I, provided $G \not\equiv 0$.

THEOREM IV. *If* $g^2G + g'h - gh' < 0$ and (y_1, z_1), (y_2, z_2) *are two independent solutions of* (1) *such that* $\phi_1 = gz_1 - hy_1$ *vanishes at least twice on* I, *then* $\phi_2 = gz_2 - hy_2$ *will vanish exactly once between two consecutive zeros of* ϕ_1, $x_1 < x_2$, *and conversely.*

Proof: ϕ_2 cannot vanish at x_1 or x_2, on account of $g^2 + h^2 > 0$ on I. If ϕ_2 does not vanish on $x_1 < x < x_2$, then ϕ_1/ϕ_2 is a continuous function with the bounded integrable derivative almost everywhere on (x_1, x_2),

$$(\phi_1/\phi_2)' = [gh]W[y_1, y_2]/\phi_2^2 = [gh](y_1z_2 - y_2z_1)/\phi_2^2$$

where W is the Wronskian of the two independent solutions (y_1, z_1), (y_2, z_2) and never vanishes. Then

$$\phi_1/\phi_2 = \int_{x_1}^{x} [gh]W[y_1, y_2]/\phi_2^2 dt$$

is a monotonic function, not a constant, and vanishes at $x = x_1$. This contradicts $\phi_1(x_2) = 0$. Hence ϕ_2 vanishes at least once on $x_1 < x < x_2$. By symmetry ϕ_1 vanishes at least once between two consecutive zeros of ϕ_2. Hence each vanishes exactly once between two consecutive zeros of the other and the zeros are said to separate.

If $g \equiv 0$, $h \equiv 1$, we have that the zeros of two linearly independent solutions, y_1, y_2 separate each other.

If $g \equiv 1$, $h \equiv 0$, and (y_1, z_1) and (y_2, z_2) are linearly independent, the zeros of z_1 and z_2 separate each other if $G < 0$.

THEOREM V. *If* $\theta_1 = g_1z - h_1y$, $\theta_2 = g_2z - h_2y$, *where* $g_1h_2 - g_2h_1 \not\equiv 0$, $g_1^2G + g_1'h_1 - g_1h_1' < 0$, $g_2^2G + g_2'h_2 - g_2h_2' < 0$, *and if* θ_1 *and* θ_2 *vanish at least twice on* I, *their zeros separate each other on* I.

Proof: Without loss of generality we take $g_1h_2 - g_2h_1 \equiv 1$, and note that if $x_1 < x_2$ are two consecutive zeros of θ_1, $\theta_2(x_1) \neq 0$ and $\theta_2(x_2) \neq 0$. If $\theta_2 \not\equiv 0$ on $x_1 < x < x_2$, consider the continuous function θ_1/θ_2 which has the bounded integrable derivative almost everywhere on (x_1, x_2),

$$\begin{aligned}(\theta_1/\theta_2)' &= -[g_1h_1] - 2(g_1h_2' - g_2h_1' + h_1h_2K - g_1g_2G)\theta_1/\theta_2 - [g_2h_2]\theta_1^2/\theta_2^2, \\ &= Q(\theta_1/\theta_2).\end{aligned}$$

Now

$$-\theta_1/\theta_2 = \int_{x_1}^{x} Q(\theta_1/\theta_2)dt = \int_{x_1}^{x} Q(\theta_1/\theta_2)dt.$$

By hypothesis, $[g_1h_1]$ is always negative. Hence we may show that given $\epsilon > 0$, there exists a $\delta_\epsilon > 0$, such that for each x in $x_1 < x < x_1 + \delta_\epsilon$, θ_1/θ_2 is positive and numerically less than ϵ. Similarly we may show that

for each x in $x_2-\bar{\delta}_\epsilon < x < x_2$, θ_1/θ_2 is negative and numerically less than ϵ. This yields a contradiction and hence θ_2 must vanish at least once on $x_1 < x < x_2$. By symmetry, θ_1 vanishes at least once between two consecutive zeros of θ_2. Hence the zeros of θ_1 and θ_2 separate each other on I.

If $g_1h_2-g_2h_1 \equiv 0$, the zeros of θ_1 and θ_2 coincide.

If $g_1 \equiv 1$, $h_1 \equiv 0$, $g_2 \equiv 0$, $h_2 \equiv 1$, we obtain the result that the zeros of y and z separate each other on I when $G < 0$.

Theorems I–V above may be generalized to the case of an nth order linear, differential equation whose coefficients are bounded and integrable in the sense of Lebesgue. The present theorems represent extensions of the work of Sturm and his successors. See Bôcher, "Leçons sur les Méthodes de Sturm," 1917, Borel monograph, pp. 43–52.

CONCERNING IRREDUCIBLY CONNECTED SETS AND IRREDUCIBLE CONTINUA[1]

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1. A point set M is said to be *irreducibly connected about a point set* A, if M is connected and contains A, but contains no proper subset which is connected and contains A. So far as the author knows, no previous study has been made of irreducibly connected sets, except for the special case where the set A consists of two points.[2]

THEOREM 1. *Every bounded continuum is irreducibly connected about the set consisting of its non-cut-points.*[3]

Proof: Let M be the continuum, and A the set of non-cut-points of M. If N is any subset of M that contains no point of A, then any point P of N is a cut-point of M. Therefore, $M-P = M_1 + M_2$, where M_1 and M_2 are two mutually exclusive sets, neither containing a limit point of the other. Each of the continua $M_i + P$ $(i = 1,2)$ contains two non-cut-points of itself,[4] and, therefore, a non-cut-point of itself different from P. It can then be easily established that this point is also a non-cut-point of M. Therefore each of the sets M_i contains a point of A, that is, a point not in N.

The set M-N is a subset of M-P, and since both M_1 and M_2 contain points which are not in N, the set M-N consists of the two sets which M_1 and M_2 have in common with M-N. Therefore, M-N is disconnected, that is, any proper subset of M that contains A is disconnected. Therefore, M is irreducibly connected about A.

Note that the preceding theorem is not necessarily true for sets which are not bounded. For example, a ray (or half-line) has only one non-cut-point, and the point itself (and not the ray) is the set which is irreducibly connected about the set of non-cut-points.

THEOREM 2. *A necessary and sufficient condition that a bounded continuum M be irreducibly connected about one of its subsets B, is that B contain all the non-cut-points of M.*

Proof: Since we have proved in theorem 1 that M is irreducibly connected about A, the set of non-cut-points of M, it follows that M is irreducibly connected about any subset of M that contains A. The condition is, therefore, sufficient.

To prove the condition necessary, we shall suppose that M is irreducibly connected about a subset B which does not contain all the non-cut-points of M. Let P be a non-cut-point of M not contained in B. Then the set $M-P$ is a proper subset of M which is connected and contains B, which contradicts our assumption that M is irreducibly connected about B. The condition is, therefore, necessary.

We see from theorem 2 that the set of non-cut-points of a bounded continuum form what we shall call a *basic set* about which the continuum is irreducibly connected. That is, the continuum is irreducibly connected about this set, but not about any proper subset of it. We have shown, therefore, that in the case of a bounded continuum there is one and only one basic set about which the continuum is irreducibly connected. In the case of an unbounded continuum, there may be no basic set.

In proving the necessity of the condition of theorem 2, we made no use of our assumption concerning M, save that M was connected. We have, therefore, proved the following theorem:

THEOREM 3. *If M is a connected set (bounded or unbounded) which is irreducibly connected about one of its subsets B, then B contains all the non-cut-points of M.*

2. A point set M is said to be *an irreducible continuum about a point set A*, if M is a continuum and contains A, but contains no proper subset which is a continuum and contains A. An extensive study has been made of irreducible continua for the special case where the set A consists of two points.[5] The theorem that a bounded continuum contains an irreducible continuum about *any* subset of the continuum has recently been proved by W. A. Wilson.[6]

THEOREM 4. *Every bounded continuum is an irreducible continuum about the set consisting of its non-cut-points.*

Proof: This theorem follows from theorem 1, and the fact that if a continuum is irreducibly connected about a set A, it is an irreducible continuum about A. However, an irreducible continuum about a set A is not necessarily irreducibly connected about A.

The theorem analogous to theorem 2 is not true for irreducible continua. In fact, a bounded continuum may be an irreducible continuum about a subset that does not contain any of its non-cut-points. For example, an arc is an irreducible continuum about the set consisting of its interior points, which is identical with the set of its cut-points.

In general, a bounded continuum does not contain a unique basic set about which it is an irreducible continuum. This is evidently true even when the continuum is irreducible about a set consisting of only two points.

Theorem 2 implies that if a bounded continuum M is irreducibly connected about a set A, then A always contains a basic set about which M is irreduciby connected. But the analogue of theorem 2 is not true for irreducible continua. That is, a bounded continuum M may be an irreducible continuum about a set A, and yet A may contain no basic set about which M is an irreducible continuum. That this may be true even when A is closed may be seen from the following example.

Kuratowski[7] has given an example of an indecomposable continuum, M, which contains a subset N, every point of which can be joined to the point (0,0) by an arc lying in N, and which is such that every point of M-N is a limit point of N. If we let A be the set of points: (0,0), and $(1/3^n,0)$, for $n = 0, 1, 2, \ldots$, then any continuum in M containing A must contain N and must, therefore, be identical with M itself. Suppose, now, that A contained a basic subset A_1—which evidently must contain more than one point. Let P be any point of A_1 except (0,0). Since the set A_1-P is a proper subset of A_1, the continuum M contains a proper subcontinuum M_1 containing A_1-P. If Q is any other point of A_1, then M contains an arc M_2 joining P to Q. Since M is indecomposable, this arc is a proper subset of M. The sum of the two continua M_1 and M_2 is a continuum which contains A_1 and must, therefore, be identical with M. But this is impossible, as an indecomposable continuum cannot be expressed as the sum of two of its proper subcontinua.

The fact that A contains no basic subset also follows from theorem 5, and the fact that M is not an irreducible continuum about a set consisting of two points of N.

THEOREM 5. *If a continuum M (bounded or unbounded) is an irreducible continuum about a subset consisting of n points, but not about any subset consisting of n-1 points, and if A is a closed subset of M such that M is an irreducible continuum about A but not about any proper closed subset of A, then A consists of exactly n points.*

Proof: The theorem is evidently true for the trivial case $n = 1$, and we can, therefore, suppose in the following argument that $n \geqq 2$. Let $P = P_1 + P_2 + \ldots + P_n$ be a subset about which M is an irreducible continuum. If the theorem is not true, then A consists of at least $n + 1$ points, and at least one point of P is not a point of A.

Suppose A consists of less than $n + 1$ maximal connected sets. Then, from what we have just said, at least one of these contains more than one point, and, therefore, contains an infinite number of points. Let Q be a finite subset of A which contains every point of P in A, and at least one point of every maximal connected subset of A. Since Q is a finite set, Q is a proper closed subset of A, and, therefore, M contains a proper subcontinuum N which contains Q. Since N is a proper subcontinuum of M, it does not contain all of the points of P, and since A-Q contains no points of P, it follows that their sum $N + A$ is a proper subcontinuum of M containing A. This is evidently a contradiction of our hypothesis that M is an irreducible continuum about A.

Suppose that A consists of at least $n + 1$ maximal connected sets. Then A can be expressed as the sum of $n + 1$ closed, mutually exclusive subsets: $A_1, A_2, \ldots, A_{n+1}$. Let M_i denote a proper subcontinuum of M containing $A - A_i$. Since M_i cannot contain all the points of P, let Q_i denote those points of P which are not in M_i. Any subcontinuum containing A_i and any other point of A must contain Q_i; otherwise if this continuum be added to M_i we obtain a proper subcontinuum of M (because it does not contain all of P) which contains A, which is contrary to hypothesis.

Since the set Q_1 is not vacuous, it contains at least one point of P. Since each of the sets $M_2, M_3, \ldots, M_{n+1}$, contains A_1 and at least one other point of A, each of these sets contains Q_1 and, therefore, no point of Q_1 is a point of any of the sets: $Q_2, Q_3, \ldots, Q_{n+1}$. The set Q_2, therefore, contains at least one point of $P - Q_1$. As before, no point of Q_2 is a point of any other set Q_i, and by continuing this process, we see that the sets Q_i are mutually exclusive. Since there are $n + 1$ of these sets, and P contains only n points, at least one of these sets must be vacuous, which is impossible on account of the way in which they were selected. Therefore, theorem 5 is true.

3. In conclusion, the author wishes to point out that these theorems are true not only for plane point sets, but for sets in space of any number of dimensions.

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[1] Presented to the American Mathematical Society, Sept. 9, 1926.

[2] Knaster and Kuratowski, *Fund. Math.*, **2**, 1921 (206–255).

[3] If M is a connected point set, and P is a point of M, then if $M - P$ is not connected, P is said to be a *cut-point* of M; if $M - P$ is connected, P is said to be a *non-cut-point* of M. Moore, R. L., these PROCEEDINGS, **9**, 1923 (101–106).

[4] Moore, R. L., loc. cit., Theorem A^*.

[5] For a bibliography, see Kuratowski, C., *Fund. Math.*, **3**, 1922 (230).

[6] *Trans. Amer. Math. Soc.*, **27**, 1925 (429–440).

[7] Loc. cit. (209–210). See figure 1.

INTERPRETATION OF DEVIATIONS FROM OHM'S LAW

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Several years ago Bridgman[1,2] reported deviations of a few per cent from Ohm's law when gold or silver films were traversed by currents of several million amperes per cm.^{2} This note points out certain conclusions which logically follow if these experiments are interpreted along the lines of two familiar theories of metallic conduction.

I. *Thomson's Second (or Doublet) Theory.*—The assumptions made in this theory are (1) that atoms are permanent electric doublets of moment μ, (2) that the axes of these doublets are distributed in direction according to the laws of classical statistical mechanics, (3) that each atom discharges an electron on the average p times per second in the direction of its axis, (4) that these electrons go an average distance b before being captured by another atom. The electrical conductivity is given by

$$\sigma = \frac{Npeb}{E}\overline{\cos\theta}, \text{ where } \overline{\cos\theta} = \frac{\mu E}{3kT} - \frac{\mu^3E^3}{45k^3T^3} + \ldots\ldots,$$

where $\overline{\cos\theta}$ gives the average degree of alignment of the doublets in the direction of the applied field E. Deviations from Ohm's law, on this theory occur if the second term in the expansion of $\overline{\cos\theta}$ becomes appreciable. Putting σ for the normal conductivity and σ^1 for the conductivity when Ohm's law is departed from, we have

$$\frac{\sigma-\sigma^1}{\sigma} = \frac{\mu^2E^2}{15k^2T^2} + \ldots, \text{ whence } \mu = \frac{kT}{E}\left(15\frac{\sigma-\sigma^1}{\sigma}\right)^{1/2}. \qquad (1)$$

Bridgman found, for silver, a deviation of 1 per cent from Ohm's law at a current density of about $4.8\,(10)^6$ amperes per cm.^{2} The temperature of the film was not measured but was certainly well below the melting-point of silver, since it withstood currents considerably larger than this. We cannot be in error by a factor as large as 2 if we take T to be 500°K. Using the above current density and the resistivity of silver at this temperature we find E = 13.5 volts/cm. or 0.045 e.s.u./cm.

Substituting thus for T and E and for the Boltzmann gas constant $k = 1.346(10)^{-16}$, and putting $(\sigma-\sigma^1)/\sigma = 0.01$, equation (1) gives $\mu = 0.58(10)^{-12}$ e.s.u. cm. as the magnitude of the electric moment of the atomic doublet required to account for the observed deviations from Ohm's law. To produce such a doublet would require the preposterous concentration of 100,000 electrons at one end and an equal positive charge at the other end of the atomic diameter.

The other possible effect of an electric field on the conductivity is the bending of the path of the electron between its ejection from one atom and its absorption by another, and this would produce a deviation from Ohm's law in the reverse direction.

We conclude, therefore, that the order of magnitude of the deviations observed by Bridgman constitutes conclusive evidence against Thomson's doublet theory of metallic conduction.

II. *Classical Free Electron Theory.*—This, the most familiar of the modern theories of conduction, involves the assumption of free electrons which are acted on by the electric field during their free paths so as to give a net rate of transfer in the direction of the field. The conductivity is given by

$$\sigma = Nle^2/am\bar{S} \qquad (2)$$

where N is the number of free electrons per unit volume, e, m and l are their charge, mass and free path, $\bar{S}$ is their average speed and a is a constant which is 2 on the crude theory of equal free paths and equal speeds, but is $\pi/2$ on the more reasonable theory of speeds and free paths with normal probability distributions.

Ohm's law follows from the additional assumption that $\bar{S}$ is independent of the field. The classical theory assumes that the free electrons are in thermal equilibrium with the metal atoms, from which follow both Ohm's law and the Franz-Wiedemann ratio.

To be consistent with the assumption of free electrons subject to the action of the field, we must admit that the mean energy of the electrons, during current flow, is actually somewhat greater than that of the atoms on account of the additional energy contributed to them by the field, so that the assumption of thermal equilibrium is justifiable only as a first approximation and for small fields. It is to departures from this equilibrium that we must look to explain the deviations from Ohm's law on the free electron theory.

Actually, during current flow, the electrons must move with a *terminal speed* such that their rate of gain of energy from the field equals the rate at which they lose energy by momentum transfers at their elastic impacts with atoms. In an earlier paper[3] I gave an expression for this terminal speed, derived for the motion of electrons through gases, but directly applicable to the present case. This expression is

$$\bar{S} = \frac{1}{1.086\sqrt{m}}\left[\alpha T + (\alpha^2 T^2 + \frac{l^2E^2e^2M}{1.134m})^{1/3}\right]^{1/2} \qquad (3)$$

where $\alpha = 3k/2$, M is the atomic mass, and the numerical factors arise from the distinction between average and root mean square speeds.

Substitution of (3) into (2) gives, for the conductivity in a field E,

$$\sigma^1 = \frac{Nle^2}{a\frac{\sqrt{m}}{1.086}\left[\alpha T + (\alpha^2 T^2 + \frac{l^2E^2e^2M}{1.134m})^{1/2}\right]^{1/2}}. \quad (4)$$

Ohm's law is given by the value of σ^1 as E becomes vanishingly small, and is

$$\sigma = \frac{Nle^2}{a\frac{\sqrt{m}}{1.086}\sqrt{2\alpha T}}, \quad (5)$$

which is the classical expression.

From these two expressions, and confining ourselves to *small* deviations from Ohm's law such that $(\sigma-\sigma^1)<<\sigma$, we find

$$l = \frac{\alpha T}{Ee}\left[\frac{9.072m}{M}\left(\frac{\sigma-\sigma^1}{\sigma}\right)\right]^{1/2} \quad (6)$$

for the value of mean free path required, on the free electron theory, to account for the observed deviations from Ohm's law.

Substitution of the same experimental data as were used in connection with the Doublet Theory gives $l = 3.13(10)^{-6}$ cm. This value is a reasonable and possible one since (1) thin films begin to show abnormal conductivity for thicknesses less than about three times this value, (2) this value, in conjunction with that of the product Nl gives a value of N which is sufficiently small to make the contribution by free electrons to the specific heat relatively small, (3) it is of the order of that suggested by Bridgman[4] as most probable in the light of his "gap" theory of conduction, which is a modified free electron theory.

Incidentally this analysis suggests certain relationships which might be subjected to further experimental test. For example, it indicates deviations proportional to E^2, as already suggested on more general grounds by Maxwell. The results of Bridgman do not conform very accurately to this relation, but are not decisive since they were performed with a film temperature which increased with increase in E and the lack of conformity with predictions is in a direction to be thus explained. Also the analysis predicts that $(\sigma-\sigma^1)/\sigma$ should vary as $(l/T)^2$ at constant E, or as l^2 at constant current density. Bridgman's "gap" theory makes l vary as $1/\sqrt{T}$, so that we might look for temperature variation of $(\sigma-\sigma^1)/\sigma$ proportional to $1/T^3$ at constant e.m.f. and proportional to $1/T$ at constant current. No data are at present available to test these predictions.

Conclusions.—The above analysis simply points out certain logical consequences of two common theories of metallic conduction and shows that

experimental data definitely disprove the doublet theory but are consistent with the free electron theory. It is certain, of course, that this free electron theory can at best be but a crude approximation to the real process of conduction, and the recent quantum theory of Frenkel[5] seems to offer promise of portraying more accurately the actual phenomena involved in current transfer in metals.

[1] Bridgman, *Proc. Nat. Acad. Sci.*, **7**, 299 (1921).
[2] Bridgman, *Proc. Am. Acad. Arts Sci.*, **57**, 131 (1921).
[3] Compton, *Physic. Rev.*, **22**, 333 (1923).
[4] Bridgman, *Ibid.*, **19**, 114 (1922).
[5] Frenkel, *Zs. Physik*, **29**, 214 (1924).

RUBIDIUM AND CAESIUM-LIKE DOUBLETS OF STRIPPED ATOMS

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As in a previous paper[1] dealing with the potassium-like doublets of stripped atoms it has been found possible to apply the so-called regular and irregular doublet laws to elements in the same rows with rubidium and with caesium.

Using the data as given by Fowler,[2] Eder and Valenta,[3] Exner and Haschek,[4] corrected to vacuum when necessary, and following the method of extrapolation described in our paper on potassium, it was relatively easy to identify the ($5s$–$5p_2$), ($5s$–$5p_1$) doublets of Rb_I, Sr_{II}, Yt_{III}, and Zr_{IV}, as well as the ($6s$–$6p_2$), ($6s$–$6p_1$) doublets of Cs_I, Ba_{II}, La_{III}, Ce_{IV}, and Pr_V. In nearly all cases the location of the identified lines were predicted to within one or two A.U. of the lines given in our tables. In one case a line was found to within 0.3 A.U. of its predicted value. In no case are these lines, which are due to higher states of ionization, reported as having been found in the arc spectrum, but are recorded as relatively very intense lines in the spark spectrum.

Figure 1 shows that the frequencies of the $5s$–$5p_2$, and $6s$–$6p_2$ lines, as given in column 4 of tables 1 and 2, progress almost linearly with the atomic number, as the core charge increases.

Column 3 of tables 1 and 2 shows how beautifully the so-called regular doublet law is verified by the small but systematic variations in λ. This systematic change is better appreciated by taking first and even second differences between successive values of $\Delta\lambda$. These differences are a great help in identifying the correct lines belonging to each doublet. The values

of the screening constant, computed from the Sommerfeld regular doublet law

$$\Delta\nu = K(Z-s)^4,$$

where $K = 0.0234$ for the $5p_1\ 5p_2$ doublets and $K = 0.0135$ for the $6p_1\ 6p_2$ doublets are shown in column 6 of tables 1 and 2. These values are entirely consistent with the relative contraction of the different shells as the state of ionization increases from element to element. Also it can now be seen that the screening constant of the alkali metals from Li to Cs, each one of which is at the head of a sequence of stripped atoms, shows a surprisingly regular progression. Since the term values $5s$, $5p_2$, $6s$, $6p_2$,

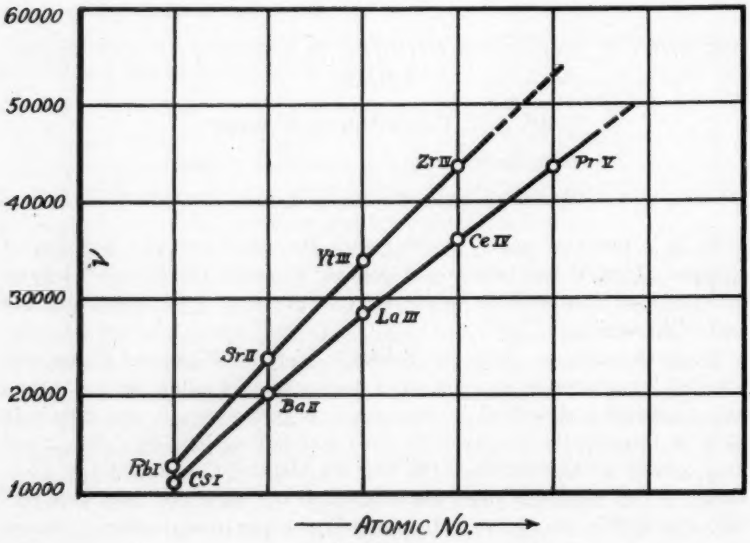

FIGURE 1

for the first two elements of each sequence are given in Fowler,[2] the term values of the rest of the elements can be obtained, as in the case of potassium,[1] from an almost linear relation between $1/q$ and the atomic number. Here again the irregular doublet law is confirmed by a very nearly constant difference between $\sqrt{\nu_s/R}$ and $\sqrt{\nu_{p_2}/R}$, throughout each sequence of stripped atoms, as shown in figures 2 and 3, and in column 7 of tables 1 and 2.

It should be pointed out that the term stripped atoms, as here used, means that the electrons in each atom of a sequence have been reduced to the same number, even though they may not come from the same shell.

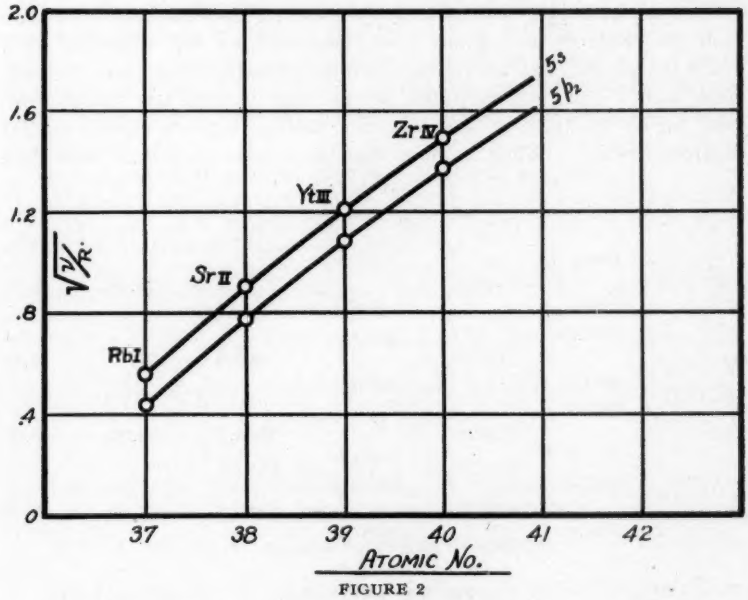

FIGURE 2

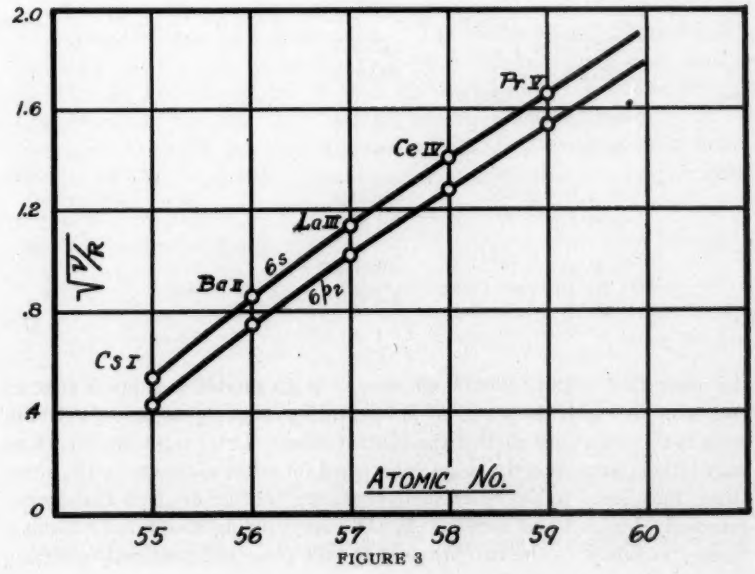

FIGURE 3

If we accept Bohr's electronic arrangements for these elements, not more than two electrons are, in any case, removed from the outermost shell, which means that any additional electrons probably come from the next inner shell. This, however, does not necessarily cause any serious difficulty for we are here, in each sequence, dealing with an exactly similar electron residium. When an atom, that has lost electrons from more than

TABLE 1

$(5s-5p_1)$ and $(5s-5p_2)$

	λ	$\Delta\lambda$	ν	$\Delta\nu$	s	$\sqrt{\nu_s}-\sqrt{\nu_{p_2}}$
	7949.8		12578.9			
Rb_I		147.4		237.6	26.96	38.25
	7802.4		12816.5			
	4216.7		23715.2			
Sr_{II}		137.8		801.5	24.40	42.83
	4078.9		24516.7			
	2946.9		33934.3			
Yt_{III}		129.1		1553.7	22.95	45.01
	2817.8		35488.0			
	2287.4		43718.3			
Zr_{IV}		123.0		2483.2	21.95	46.23
	2164.4		46201.5			

TABLE 2

$(6s-6p_1)$ and $(6s-6p_2)$

	λ	$\Delta\lambda$	ν	$\Delta\nu$	s	$\sqrt{\nu_s}-\sqrt{\nu_{p_2}}$
	8945.9		11178.3			
Cs_I		422.4		554.0	40.77	35.0
	8523.5		11732.3			
	4935.5		20261.5			
Ba_{II}		380.2		1690.9	37.19	38.3
	4555.3		21952.4			
	3518.1		28424.3			
La_{III}		345.5		3095.7	35.24	39.8
	3172.6		31520.0			
	2769.2		36111.6			
Ce_{IV}		314.1		4619.8	33.81	40.6
	2455.1		40731.4			
	2295.4		43566.1			
Pr_V		...		....	...	41.1
	....		.....			

the outer shell, regains its first electron, it is reasonable to suppose that an end orbit to which this electron jumps, thus producing radiation, exists in what is the outermost shell of the neutral atom. Later returning electrons may either jump directly to an inner shell, or what amounts to the same thing, may jump to the outer shell, replacing another electron that simultaneously jumps to an inner shell, the energy being integrated to emit a single frequency, or the two jumps may take place independently emitting

two separate frequencies. Unless we assume an entirely different arrangement for electrons in these neutral atoms it is difficult to explain in any other way the remarkable similarity between the doublets of these sequences and those of Li, Na, Cu, Ag and Au, from all of which the electrons are removed from the outermost shell in each successive stage of ionization.

[1] R. C. Gibbs and H. E. White, these PROCEEDINGS, **12**, 448 (1926).

[2] A. Fowler: *Series in Line Spectra*, pp. 105, 108, 132, 137, Fleetway Press, **1922.**

[3] J. M. Eder and E. Valenta, "Atlas Typischer Spektren;" pp. 92, 94, 96, 98, 107, *Kaiserliche Akademie der Wissenschaften*, Wien, **1911.**

[4] F. Exner und E. Haschek: *Die Specktren der Elemente*, Bd. III, pp. 20, 105, 176, 321, 324, Franz Deuticke, Leipzig und Wien, **1912.**

THE FINE STRUCTURE OF CERTAIN LINES AND ENERGY LEVELS OF CADMIUM

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From photographs of the fringe systems caused by each of two different quartz Lummer-Gehrcke plates crossed with a quartz prism or grating spectrograph the fine structures of a number of lines of a water-cooled cadmium arc have been determined. The results appear in the following table in which a component marked +0.077(2) is a component which is 0.077 A.U. away from and on the long wave-length side of the strongest component of the line observed and has an estimated intensity of 2 as compared with 10, the intensity assigned to the strongest component.

TABLE I

WAVE-LENGTH	LINE	STRUCTURE			
5085.9	$2p_1-2s^{(2)}$	+0.077 (7);	0.000 (10);	−0.025 (4)	
4799.9	$2p_2-2s^{(2)}$	+0.058 (6);	0.000 (10);	−0.034 (3);	−0.081 (3)
4678.2	$2p_3-2s^{(2)}$	+0.031 (8);	0.000 (10);	−0.056 (4)	
3252.5	$2p_1-3s$	+0.031 (2);	0.000 (10);	−0.010 (3)	
3133.2	$2p_2-3s$	+0.024 (1);	0.000 (10);	−0.012 (1);	−0.033 (1)
3080.9	$2p_3-3s$	+0.013 (3);	0.000 (10);	−0.023 (2)	
3614.4	$2p_1-3d''$	+0.037 (2);	0.000 (10);	−0.023 (4)	
3610.5	$2p_1-3d$		0.000 (10);	−0.036 (3)	
3466.2	$2p_2-3d'$	+0.031 (3);	0.000 (10);	−0.015 (5)	
3403.6	$2p_2-3d''$	+0.017 (5);	0.000 (10)		
2980.6	$2p_1-4d$		0.000 (10);	−0.026 (1)	
2881.2	$2p_2-4d''$	+0.021 (1);	0.000 (10);	−0.010 (5)	
2836.9	$2p_3-4d''$	+0.011 (4);	0.000 (10)		
4413.1	$2p_2-3S$	+0.028 (1);	0.000 (10);	−0.015 (3)	
3499.9	$2p_3-3D$	+0.017 (2);	0.000 (10);	−0.010 (4)	

The positions of the components as recorded are probably correct to 0.001 A.U. Care was taken so as not to mistake the two parts of a self-reversed component for two separate components. The fine structure of these lines and the fine structure of several others reported by other observers can be accounted for by assigning certain fine structures to the lower energy levels of cadmium.

In a paper which has been submitted to the *Philosophical Magazine* for publication the experimental work is discussed giving methods and indications of the probable accuracy of the results and also a diagram of the lower energy levels is given showing the fine structure of the levels and the origin of each component of all the lines of cadmium whose fine structure is known.

[1] NATIONAL RESEARCH FELLOW IN PHYSICS.

[2] The fine structure of these lines has been recorded by several other observers. Janicki, L., *Ann. Physik*, **19**, 1906 (36–79); *Ibid.*, **29**, 1909 (833–868); Wali-Mohammad, Ch., *Ibid.*, **39**, 1912 (225–250); Takamine, *Proc. Tokyo Math. Phys. Soc.*, **8**, 1915 (51). No two of these observers, including the writer, disagree on the position of a component by more than 0.001 A.U.

ON THE SPECTRUM OF ARGON

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The structure of the spectrum of the normal Argon atom has been sought for by several investigators, but the problem is a difficult one on account of the complexity of the spectrum. Last year Professor Lyman and the writer[1] published the wave-lengths of the "resonance" lines of Argon in the extreme ultra-violet, and these furnish a key to the solution, following the general plan of the spectrum of Neon, which is similar and has been completely worked out by Paschen. This subject has been pursued as occasion permitted, and this spring an abstract of the conclusions reached at that time was published.[2] Recently K. W. Meissner has given a grouping of the red and ultra-red Argon lines.[3]

During the past year we have obtained photographs of this spectrum in the extreme ultra-violet and throughout the ordinary part of the range with large prism spectrographs of glass and quartz from low-current arcs and various vacuum-tube discharges. The results from all these plates have yielded a large number of new lines and of combinations. Certain series can be given with some assurance. As there will not be opportunity to return to this work for some time, it seems best to publish what has been obtained so that others may benefit by it. The present note, then, gives

the more interesting results. The notation used throughout is that of Paschen for Neon.

The Principal Series.—Meissner's arrangement of the red lines is evidently in the main correct, but his $2p_4$ and $2p_3$ combinations it seems more logical to call $2p_5$ and $2p_4$, respectively. This brings their values of the inner quantum number j and their combinatory properties into better harmony with the corresponding terms in Neon. His values for $2p_1$, $2p_2$ and $2p_5$ I have not been able to verify.

As many as five members of these series have been found in one case, and this enables one to find the limits of the series with a precision of perhaps 5 units of wave-number. These series terms are identified by their constant-frequency differences with parallel series; thus, the series $1s_2-mp_4$, $1s_4-mp_4$ and $1s_5-mp_4$ will have the constant frequency differences characteristic of the terms $1s_2$, $1s_4$ and $1s_5$. In all cases the agreements were considered unsatisfactory and probably accidental if the experimental error exceeded a whole unit of wave-number; usually it was less than half of this amount.

The $1s_2$, $1s_3$, $1s_4$ and $1s_5$ terms of Argon are, respectively, 31,713.9; 32,560.1; 33,363.1; 33,970.0 to within a few units. These combine with a large number of $2p$ terms, whose values are given in the following table 1.

TABLE 1

PRINCIPAL SERIES TERMS

m =	2	3	4	5	6
mp_{10}	23017.7	10453.2	6044.9	3941.3	....
	(4.5)	(2.3.4.5)	(2.3.4)	(2.3.4)	....
mp_9	21651.0	10171.0	5948.7	3908.8	....
	(5)	(5)	(5)	(5)	....
mp_8	21496.4	10113.9	5922.0	3893.8	....
	(4.5)	(2.4.5)	(2.4.5)	(5)	....
mp_7	21026.5	9961.8	5856.2	3858.5	....
	(2.3.4.5)	(2.3.4.5)	(4.5)	(4.5)	....
mp_6	20876.2	9930.0	5843.2	3852.2	2732.1
	(2.4.5)	(2.4.5)	(2.4.5)	(4.5)	(5)
mp_5	19982.0	8653.6	4512.1(?)	....	....
	(2.3.4.5)	(2.3.4.5)	(2.4.5)	....	....
mp_4	19824.0	8644.1	4478.3	2455.0	....
	(2.4.5)	(2.4.5)	(2.4.5)	(2.4)	....
mp_3	18391.0	?	.....	....	....
	(2.4)	.....	.....	....	....
mp_2	19617.3	?	.....	....	....
	(2.3.4.5)	.....	.....	....	....
mp_1	16132.8(?)	8706.2	.....	....	....
	(2.3?.4)	(2.4)	.....	....	....

(2.4.5) written under a term indicates that this term combines with $1s_2$, $1s_4$ and $1s_5$.

Of these principal series, the higher five should, by analogy to Neon, run to the same limits, and appear to do so, in a Ritzian manner. The

series $1s-mp_4$ and $1s-mp_5$ should run to other limits, which appear to lie some 1400 units higher than the normal ones, but this value cannot as yet be properly fixed. If so, this number takes the place of the number 780 in Neon, and has to be added to these series to make them Ritzian.

Ionization Potential of Argon.—The values of the various $1s$ terms have thus been determined from the series given above. The strong fundamental pair of lines in the extreme ultra-violet then furnish the term $1p$, which should correspond to the ionization potential of the element. This works out to be 15.69 volts, which is to be compared with the value of 15.3 volts given by Hertz. A second potential should also exist if the above work is correct, at a value 1400 units of wave-number higher, which brings it to 15.86 volts.

Subordinate Series.—Over eighty combinations of the $2p-md$ and $2p-ms$ classes have been found and are presented in table 2. This table shows the actual values of the terms obtained; that is, if each number is subtracted from the p term at the top of its column, the result is an observed wave-number. The agreement among the numbers in a horizontal row indicates the size of the experimental error. Of all these d and s terms many series must be formed. The only one that seems certain enough to be worth mentioning is made of the terms 7430.7; 4674.0; 3210.8 and 2342.3

TABLE 2

SUBORDINATE SERIES TERMS

$2p_{10}$	$2p_9$	$2p_8$	$2p_7$	$2p_6$	$2p_5$	$2p_4$	$2p_3$
7430.6	7430.7	7430.5	7430.9	7431.9	....	....	....
7353.5	....	7353.2	7353.9	....	....	....	....
....	6360.7	6359.6(?)	....	6361.4	....	6361.2	....
6068.5(?)	6069.9(?)	6071.0(?)	6071.3	6071.6	....	....	....
6016.8	....	....	6017.2	....	6016.6	....	....
5952.7	....	5952.3	5952.0	5951.4	5952.1	5952.4	5952.3
5319.9	5320.2	....	5319.5	....	....	....	....
5181.8	....	5181.1	5180.9	5180.8	....	....	....
5027.1	5026.9	....	5026.3	5026.6	5026.8	5026.8	....
....	4953.7	4953.8	....	4953.5	....	....	....
....	4784.0	....	4784.3	4784.0	....	....	....
4673.8	4674.1	4674.3	....	4673.5	....	....	....
4634.9	....	4633.3	4634.7	4634.0	4633.9	....	....
....	3741.1	....	....	3740.7	3739.8	3740.7	3740.1
3646.3	....	....	....	3646.1	....	....	....
3609.1	....	3609.5	3608.0	....	3608.6	3608.6	....
....	3557.0	3556.9	....	3556.4	....	3556.4	3556.7
3495.8	3496.2(?)	....	....	....	....	....	....
....	....	3306.3	....	3306.3	....	3306.9	....
3210.7	3211.3	....	....	3210.7	....	3210.5	....
2563.7	....	2564.8	....	....	....	2564.0	....
2342.4	....	....	....	2342.2	....	....	....

The lines marked with a query appear to take part in two different combinations, or else the agreement is not quite satisfactory.

in succession and appears to be of the diffuse type, though it may be the s_8 series.

Observations in the Extreme Ultra-Violet.—The measurements in the extreme ultra-violet have yielded the lines shown in table 3. Several of these are marked with a query because they have been observed relatively few times, or because they may belong to some impurity. The column of terms is found by subtracting the observed wave-numbers from 127,109, the value of $1p$.

TABLE 3

ULTRA-VIOLET LINES

INTENSITY	WAVE-LENGTH	WAVE-NUMBER (VAC.)	TERM
9	1066.70	93747	...
10	1048.26	95394	...
2	946.95	105603	21506
2	941.39	106226	20883
4	932.09	107286	19823
1−(?)	931.43	107362	...
1−(?)	920.43	108645	...
5	919.80	108719	18390
1−(?)	919.17	108794	...
1+(?)	918.70	108849	...
1−(?)	908.48	110074	...
1−(?)	900.18	111089	...
2	887.43	112685	...
1+	883.20	113225	...
1	875.54	114216	...
1	871.10	114797	...
1	863.64	115793	...
1−	850.64	117559	9550
1	848.71	117826	...

New Combinations in Argon and Neon.—It is noteworthy that we have in table 3 several lines which are combinations between $1p$ and various $2p$ terms. In the column of terms we see 21,506, which is near the value of $2p_8$ in table 1; then 20,885, near $2p_6$; 19,825 near $2p_4$ and 18,390 near $2p_3$. It is to be remembered that the wave-lengths may not be accurate to 1 part in 10,000, and thus an error of 10 wave-number units could easily arise. The number of these combinations seems to make them reasonably certain. They violate the rules of selection and involve a change of j by 2 units for combinations between $1p$ and $2p_4$, $2p_6$, $2p_8$ and of 0 for the perhaps doubtful combination with $2p_3$.

These considerations are supported by the discovery of some faint lines in the Neon spectrum which seem to have the same cause. They are λλ 667.46; 665.30; 662.84. Their wave-numbers are 149,822; 150,308; 150,866, whereas the combinations should yield (Paschen's data) with $2p_8$ 149,825; with $2p_6$ 150,316; and with $2p_4$ 150,859. The disagreement is well

within the experimental error. Since the p_3 combination was not found in Neon, its apparent occurrence in Argon may be doubtful. Combinations of $1p$ with the higher terms belonging to mp_4, mp_6 and mp_8 were looked for, but in vain, both in Argon and Neon. Combinations of $2p_1$ and $2p_3$ with mp_4, etc., were likewise absent. One curious term was found and is mentioned in table 3. Its value is 9550 (better value probably 9550.7). This same term (if it be the same) combines with $1s_2$ and $1s_4$ strongly, and with $1s_5$ faintly, and it can be arranged in a very proper series with other terms as follows: 9550.7; 5643.7; 3728.5; and perhaps 2673.5. All of these, after the first, combine with $1s_2$ and $1s_4$, and hence appear to form a series of the $1s-mp_4$ type. If, however, the faint $1s_5$ combination is accidental, the type would then become $1s-mp_3$. In any case this series appears to converge at a point not far from where the other series do, but its first term is obscure and I have not been able to fit it into table 1 without introducing violent and improbable variations in intensity in successive series members.

There is some evidence for the existence of principal series in Argon in which the combinations of mp are with $1s_2$, $1s_3$ and $1s_4$. Such a term is given in table 1 as $2p_1$. If correct, it will involve a transition in j from 0 to 0 which is usually forbidden. Further observational material will be needed to clear up this point.

[1] *Nature*, **116**, 358 (1925).
[2] *Physic. Rev.*, **27**, 799 (1926).
[3] *Zeits. Physik*, **37**, 238 (1926).

IONIZATION IN REACTING GASES

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It has been known for years that the variation of the rate of thermionic emission and of chemical action with temperature follow the same general law. Chemists express the rate of chemical action by an equation of the form

$$\partial C/\partial t = AT^{1/2}e^{-E/kT}$$

while the Richardson equation

$$i = AT^{1/2}e^{-b/T}$$

fits the rate of thermionic emission for either positive or negative ions equally well.

This similarity of the two processes has led some physicists to the con-

clusion that thermionic emission is a result of chemical action between the emitter and the gas present. The values of the rate constants are also of the same order of magnitude, although, in general, b is from two to four times E/k. Richardson[1] has shown that the b obtained from the emission of electrons from platinum and from tungsten represents a greater energy change than that of any known reaction of these metals and so concludes that the emission cannot be due to chemical action.

The present research indicates that temperature and the change in nature of the surface have the same effect on the liberation of ions by chemical action as by pure thermionic emission.

The apparatus and technique used were the same as described before.[2]

The data obtained from the ionic emission during chemical reactions are given in table 1.

TABLE 1

REACTION OXIDATION OF	ELECTRODE	CHARGE ON ION	T ABSOLUTE	∂T	b ($\times 10^4$)	∂b	d ($\times 10^{-8}$)
Xylene	Au	+	665	...	0.577	..	7.8
		—	665	0	0.577	0	7.8
Toluene	Au	+	694	...	0.93	..	4.47
		—	694	0	0.93	0	4.47
Benzene	Au	+	699	...	1.13	..	3.69
		—	699	0	1.13	0	3.69
Ethyl Alcohol	Au	+	650	...	1.36	..	3.06
		—	650	0	1.36	0	3.06
Acetone	Au	+	637	...	2.86	..	1.45
		—	674	37	2.70	0.16	1.54
n-Butyl Alcohol	Au	+	610	...	1.87	..	2.22
		—	652	42	1.43	0.44	2.92
n-Propyl Alcohol	Au	+	612	...	2.05	..	2.03
		—	661	49	1.47	0.57	2.92
Gasoline	Au	+	655	...	2.65	..	1.57
		—	722	67	1.91	0.74	2.08
Hexane	Au	+	614	...	2.35	..	1.77
		—	702	88	1.53	0.81	2.71
Ethyl Ether	Au	+	641	...	3.04	..	1.37
		—	713	72	2.19	0.85	1.9
Amyl Alcohol	Au	+	569	...	3.28	..	1.27
		—	677	108	2.33	0.95	1.79
Hydrogen	Au	+	681	...	3.59	..	1.16
		—	717	36	2.14	1.19	1.74
Hydrogen	CuO	+	752	...	4.66	..	0.89
		—	750	2	5.58	0.92	0.75
Ethyl Alcohol	CuO	+	758	...	5.31	..	0.78
		—	580	178	1.43	3.88	2.91

The electrode material refers to the hot outer electrode at which the reaction and emission took place. (The oxidations listed are all surface reactions.) The cool electrode was either gold or aluminum. A potential difference between the electrodes of 225 volts was maintained by Bur-

gess B batteries. The total emitting surface was 80 sq. cm. The temperature given is that at which the current reached a value of 1×10^{-15} amps. per sq. cm.

In order to obtain an accurate measure of the temperature for the various emission currents it was necessary to have the reacting gases so attenuated that the heat of reaction did not materially affect the temperature of the electrodes. This made the study very difficult for gasoline, hexane, acetone and ether and may introduce a slight error in these cases.

Since both thermionic emission and chemical action follow the same law ($\log i - {}^1/_2 \log T$) can be plotted against $1/T$ as is customary for thermionic emission. All points fall on a straight line within the limits of experimental error. The first few points, where the emission is just detectable, usually fall beneath the line. This may be due to traces of easily oxidizable impurities in the vapor, or to the failure of the electrometer to respond to such small currents.

The values of b obtained from the slope of the line may be looked upon as a comparative measure of the various rates of reaction. Specifically b is a measure of the work done in moving an ion against its electrostatic image attraction from some point near the surface to a point where the image force becomes negligible, about 10^{-4} cm. from the surface.

A survey of the data shows a distinct interdependence between the rate and temperature of emission as given by b and T. In every case where the rate of emission is the same for the positive and negative ions, the temperature of emission is likewise the same. In general an increase in the rate of emission (∂b) of the positive ions over the negative is accompanied by an increase in the difference in emission temperatures (∂T).

Hydrogen furnishes the pronounced exception to this rule, but hydrogen will not react with oxygen except at temperatures where a thermionic emission is detectable, so in this case the reaction emission is complicated by the presence of a thermionic emission. In no other instance were readings made at temperatures at which an emission current could be obtained from either gas separately.

It will be noted that when the value of b is increased beyond a certain point, designated in ethyl alcohol "as near the limit," a difference in the temperature and rate of emission of the oppositely charged ions becomes apparent. This difference must either result from a separate mechanism for the emission of the oppositely charged ions or be due to the presence of a selective force at the surface, which allows a more ready escape of ions of one sign than of the other when b is large. The interdependence of (∂T) and (∂b) makes the assumption of a separate mechanism highly improbable. On the other hand this close relationship furnishes an entirely new method for studying the forces that determine contact difference of potential.

As stated before, b is a measure of the work done in removing an ion from some point near the surface to a point outside the sphere of electrostatic image attraction. This work is given by the expression $w = bk$, where k is Boltzman's constant. The electrostatic image attraction is given by $F = e^2/4x^2$, where x is the distance of the ion from the surface, and e is the charge on the ion. The work done in removing an ion from the plane $x = 0$ would be infinite, but owing to the finite dimensions of the ion there is always a lower limit to the value of x; let this limit be denoted by d. Then the work done in removing an ion when it is situated at the distance d from the surface is given by

$$w = 1/4 \int_d^\infty e^2/x^2 dx = 1/4e^2/d.$$

We have seen that the observed work is given by $w = bk$, whence

$$d = e^2/4bk.$$

Thus, given the value of b, it is possible to evaluate the distance from the surface at which the ion started when it is thrown out by the processes of emission.

The values of d are calculated on the assumption that the ions are singly charged; this may not always be justifiable. Also it is assumed that the charge on the ion is located at a discrete point and not distributed over the entire volume, and that the porosity of the surface is small as compared to the values of d obtained. It is further assumed that all the molecules at the surface are ionized by the combined image and contact forces. In the calculations b is considered to represent only the work necessary to get an ion away from the surface and does not include the energy of dissociation.

The data indicate that, when the ions are ejected from a point within a certain specific distance from the surface, a contact force causing a preferential ejection of the positive over the negative ions from the gold surface always exists; outside of this distance the contact force has no effect. A comparison of the reactions of a similar nature, like the homologous series of alcohols, suggests that this force increases rapidly upon approaching the surface.

This preferential behavior is a direct indication that the contact force is due to the presence of a region of positive electrification at the surface, appreciable, for the gold, out to about 3×10^{-8} cm. from the surface. Such a positive field tends to repel the positive ions and retard the emission of the negative ions, thus giving the resulting relationship between (∂T) and (∂b). The presence of such a field extending out from the surface is to be expected from atomic theory considerations. Since the negative elec-

tricity of the atom is not continuous but located at discrete points within the atom, it follows that the positive field of the nucleus would leak through this electron lattice and give rise to a resultant positive field for a short distance outside the electron shell.

Since only the image force was taken into consideration in calculating d, all values given for d which fall inside the region in which the contact field operates are in error to the extent to which this field is effective.

A study of the thermionic emission from several different metals in the presence of various gases has shown that the contact field is always positive for the metal surfaces. The accumulation of oxide on the surface results in a decrease in the work function for the negative ions and a simultaneous increase for the positive ions. From this it appears that the presence of the oxide on the surface brings about a weakening of the positive contact field; on highly oxidized iron surfaces the presence of a negative contact field was actually shown to exist. This change in sign of the contact field is doubtless due to the accumulation of the negative oxygen ions of the oxide on the surface.

An idea of the nature of the ions emitted is furnished by the oxidation of ethyl alcohol on copper oxide. In this case the positive ions were shown not to come from the reaction but to be thermions from the oxygen. The negative ions were emitted from the reaction just as on the gold, only at a slightly lower temperature. This indicates that the positive ions must be H^+ which could not be expected to escape from an oxide surface covered with negative oxygen ions. On the other hand $C_2H_5O^-$ ions would be repelled by the negative oxide surface, thus giving a lower temperature for the escape of the negative ions on the oxide than on the gold surface.

The mechanism of chemical action suggested by this research is that the gas molecules are ionized at the surface by the combined image and contact forces. As in the case of thermionic emission, these ions have a Maxwell distribution of kinetic energy. An ion with a sufficient energy component perpendicular to the surface will move out to a region of weak surface forces where it is possible for it to combine with an ion of opposite sign to form a neutral molecule, which is then able to escape from the surface region. Since an ion appears to require a smaller energy component to react chemically than to be emitted thermally, the fact that E/k is slightly smaller than b follows directly from Maxwell's law.

The writer is deeply indebted to Professor R. A. Millikan for his interest and many valuable suggestions.

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[1] "Emission of Electricity from Hot Bodies" by O. W. Richardson.

[2] A. Keith Brewer, *Physic. Rev.*, **26**, 633–642 (1925).

LOCALIZATION OF FORMATIVE AGENCIES IN HYDRA

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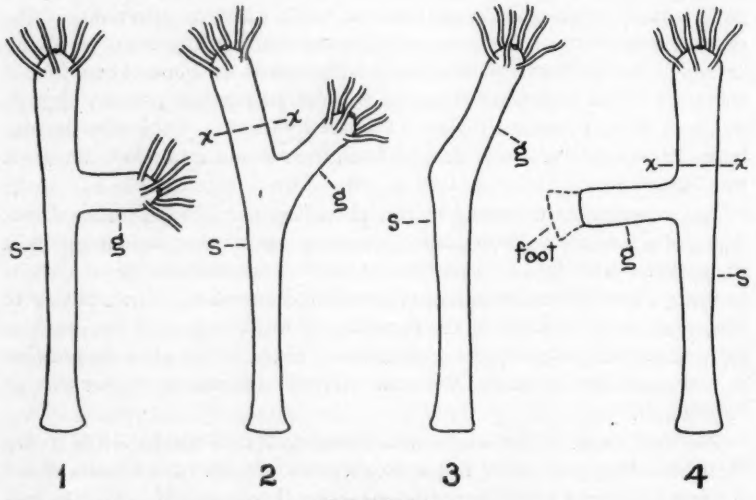
In earlier work on Hydra (Rand[1]), the regulatory behavior of complex forms produced by grafting afforded much evidence in support of the idea that agencies localized at the oral end of the animal somehow determine and control the form assumed by the column substance. In a communication published in 1911 (Rand[2]), this idea of centralized formative agencies in Hydra was briefly stated. E. N. Browne's[3] experimentation on Hydra gave a most striking demonstration of this formative dominance. She showed that a very small portion of peristome tissue grafted into the column results in the development of a lateral hydranth at the place of insertion of the graft. The transplanted tissue, without perceptible growth, appears as a part of the peristome region of the new hydranth. Otherwise the new hydranth consists of tissue derived from the column into which the graft was inserted.

The experiments described in this present paper afford additional evidence of a formative dominance of the same sort as that demonstrated in the earlier work—that is, a positive or constructive dominance which acts to bring about the development of specialized structures. In addition to this, there is given proof of the operation of a dominance of the opposite sort, manifesting itself by the retardation or even the complete suppression of a regenerative process. We shall refer to dominance of this sort as "inhibition."

The work of E. N. Browne[3] was confined to Hydra viridis. The hydra most abundantly available in our locality is a grayish brown hydra which accords fairly well with description given for H. oligactis (fusca). It was, therefore, desirable first to ascertain whether transplanted peristome tissue of this brown hydra possesses a positively acting dominance similar to that of Hydra viridis. To this end experiments similar to Miss Browne's were performed upon the brown hydra by D. E. Minnich. A small piece of peristome tissue bearing a single tentacle was transplanted into the side of the column. Fourteen such operations were made. In most of them peristome tissue from one individual was transferred to the column of another. In a few cases the graft was autoplastic. The source of the graft seemed not to affect the outcome of the experiment. The graft was inserted near the middle of the column except that in one case it was very near the oral end. The results briefly summarized were as follows.

In two cases the grafted tissue was "absorbed"—that is, it diminished in bulk day by day until it was eventually unrecognizable. There was no

external evidence of disintegration. In a third case the graft either dropped off or was absorbed. In two cases a hydranth developed at the place of insertion of the graft but the subsequent behavior of the hydranth was like that of a bud, suggesting that the graft may have been inserted at the seat of an incipient bud. In one case there was a slight outgrowth at the point of grafting, probably an abortive hydranth, but no new tentacles developed. In one case the graft was inserted very near the oral end of the column which had been beheaded. The tentacle which marked the location of the graft gradually shifted oralwards and finally joined the regenerated oral circlet of tentacles. In the remaining seven cases the grafts induced the formation of hydranths which clearly were not buds. They gave no indication



FIGURES 1–4
S, stock; *g*, graft.

of constricting off from the column in the manner of buds. In our brown hydra, therefore, just as in Hydra viridis, peristome tissue exerts a stimulating or positive form-creating dominance over column tissue.

For demonstration of inhibitory dominance, the operation was as follows. The "head" and a greater or less extent of the adjoining region of the column of one hydra was grafted laterally into the column of another (hereafter designated as the "stock") at or somewhat above its middle region (Fig. 1). Sufficient time (a day or two) was allowed for the graft to become well joined to the stock. The immediate result was a compound animal with bifurcated column and two heads (Fig. 2). Then the stock head, along with more or less of the adjoining region of column was excised,

leaving a greater or less stump projecting above the level of insertion of the graft (Fig. 2, x–x).

Appreciation of the significance of the results of this operation depends upon recognition of the fact that capacity for complete regeneration of the specialized oral parts resides at all transverse levels of the column, provided that the size of the fraction of column be in excess of a certain minimum, estimated by Peebles[4] to be from 0.005 to 0.01 of the bulk of the entire hydra (viridis).

After operations of the kind just described, the stock may or may not regenerate a head. Of eighteen such cases, in eight there was regeneration and in ten there was not. When regeneration does not take place, the healed cut end of the stump becomes rounded, appears dense and opaque, does not expand freely and day by day diminishes in bulk. There is no evidence of external loss of material. For convenience we designate the process as "absorption." Except in cases where the premature death of the compound takes place, the absorption goes on to the complete obliteration of the projecting stump of the cut stock. Meanwhile, little by little, the axis of the graft swings around into alignment with the axis of the stock. Ultimately the graft, in effect, takes the place of the missing portion of the stock and the compound becomes indistinguishable from a normal hydra. Figure 3 represents a case which is near the end of the regulatory process.

Why, in ten of the eighteen graft compounds, did regeneration of the stock head fail? In earlier work by the senior author[1] many instances of failure of regeneration under circumstances similar to those of these experiments have been observed. It cannot be too strongly emphasized that in all of these graft compounds the wound surface produced by the secondary operation of beheading the stock is one at which, in absence of any other operative complication, regeneration would certainly take place.

Is the failure due to mutilation or to the severity and shock of the operation? In the first place we have no reason to believe that the injury was less severe in the eight cases that did regenerate than in the ten that did not. Secondly, any one familiar with operations on Hydra knows that extensive crushing of a region of column or the most severe mutilation possible, short of one causing death, does not prevent regeneration. The failure in these ten cases was not due to severity of the operation.

Neither is the failure to regenerate due to disease. The absorption of the non-regenerating stump of the stock regularly takes place in compounds whose tissues appear in every way to be healthy and normal. Often absorption and active development of buds go on simultaneously in the same compound. Hydras kept under laboratory conditions are susceptible to a disease apparently due to bacterial infestation. Such individuals lose mobility and suffer a histolysis which progresses from the

tips of the tentacles basalwards and from the oral end of the column footwards. This histolysis is externally evident. The degenerating tissue is sloughed off. If such an individual is beheaded, regeneration is retarded, incomplete or fails entirely, depending on the severity or stage of the disease. If diseased individuals are used for the grafting operation or if the compound becomes diseased early in the experiment, not only does regeneration fail but absorption likewise does not take place. We therefore distinguish sharply between tissue which is passively undergoing histolysis due to disease and tissue which is undergoing a process of regulatory absorption. What we call absorption takes place at a rapid rate and passes on to completion only in individuals or compounds whose tissues otherwise, and whose activities, give evidence of healthy and vigorous conditions.

It seems a fair conclusion that the failure to regenerate is somehow due to the graft. The question then arises; will a graft of *any* sort, under appropriate conditions, inhibit regeneration?

Seven operations were made (by J. F. Bovard) in which the graft consisted of a portion of column from the aboral region of the hydra. These experiments, in respect of such details as length of graft, position of attachment of graft to stock and length of projecting stump of stock were as nearly as possible similar to those of the ten cases in which regeneration of the stock failed. In six cases the graft was the extreme aboral fourth or third of the column; in the seventh case, it was a short piece of column from the budding zone. In five cases, including the one last mentioned, the graft was attached by its oral cut surface to the stock and from the freely projecting foot-end the actual glandular foot was excised (Fig. 4). In one case the graft was attached by its foot-end (the glandular extremity having been excised) to the stock, with cut oral end free. In the seventh case the polarity of the graft was uncertain. The stock was then beheaded (Fig. 4, x–x) as in the experiments already described. In all seven cases the stock regenerated promptly, rapidly and completely. In the one case in which the graft was attached by its foot-end, the free (oral) end regenerated a head. The graft of doubtful polarity did not develop a head and gave some indication of the formation of a glandular foot at the free end.

So far as our observations have gone, a graft which is headless and, because of its polar relations to the stock, incapable of forming a head never interferes with regeneration of the stock. In earlier work by the senior author were observed cases in which, in operations of this same general sort, the graft was headless but attached by aboral end and with cut oral end free. The free end, therefore, was one possessing head potentiality. In such cases regeneration may take place at the oral cut surfaces of both graft and stock, or it may occur at either one. In the latter event the other projecting region of the column will be absorbed. Altogether, then, the experiments indicate that, in operations of the sort under consideration,

regeneration is inhibited and absorption takes place only in relation to a region of column which bears its original head or one which is capable of regenerating a head at a free oral cut surface.

The question now remains, why, in presence of a head-bearing graft, does regeneration of the stock sometimes take place, sometimes not? Analysis of the eighteen experiments reveals that the operation varied considerably as regards (*a*) the length of the graft, (*b*) the length of the projecting stump left after removal of the stock head, and, therefore, (*c*) the distance, measured along the axes, from the graft head to the cut end of the stock stump. Comparing the eight cases of regeneration with the ten cases of inhibition, it is at once evident that in the former the graft is relatively long (one-third to one-half total column length or more), the stump is long (one-fourth to one-third column length) and the axial distance between the graft head and the cut surface on the stock ranged from one-third to three-fourths column length or, in two cases, more than three-fourths. In the cases of inhibition the graft was relatively short (between one-sixth and one-half column length), the stump was short (its length not exceeding the diameter of the column when in a state of average extension) and the axial distance between graft head and wound surface ranged, one case being excepted, between one-fourth and one-half column length. In the exceptional case, the distance approximated total column length. Neither regeneration nor absorption occurred and on the ninth day of the experiment evidences of histolysis were seen. The compound died three days later. In two cases, not included in the total eighteen just referred to, the over-all axial distance between graft head and wound surface of beheaded stock was about three-fourths column length and the stock stump was about one-fourth column length. In both cases the stock regenerated. The regenerated head was then removed so as to leave the stock stump much shorter. In one case no regeneration took place; in the other, regeneration occurred but much retarded (three short tentacles appeared on the sixth day). Taken in conjunction with the fact that inhibition occurs only in relation to a head-bearing graft, these facts suggest that the inhibition is specifically due to the head or to something localized in it or produced by it, and that this inhibitory agency's effectiveness at the wound region is dependent upon such factors as length of graft, length of projecting stump of stock, or axial distance between head and wound—that is, dependent upon the space relations of head and wound.

These data, while manifestly in need of corroboration and extension, yet justify the following tentative statement. The specialized oral region ("head") of Hydra is the seat of formative agencies which may act (1) to initiate or control the development of structures appropriate in relation to itself, or (2) to inhibit the operation of a developmental mechanism where such operation would result in structure inconsistent with the at-

tainment of a form normal in relation to the dominant head region. In connection with this inhibitory effect a tissue mass which occupies an abnormal space relation to the dominant head region may undergo a regulatory reduction or "adsorption." The degree of effectiveness, at any given region, of the head-localized formative agencies depends in some way upon the space relations existing between that region and the head. Experimental data strengthening and extending these conclusions will be reported in further papers.

[1] Rand, Herbert W., 1900. "The Regulation of Graft Abnormalities in Hydra," *Arch. Entwickelungsmech.*, **9**, 161–214, pl. 5–7.

[2] Rand, Herbert W., 1911. "The Problem of Form in Hydra," *Science*, N. S., **33**, p. 391.

[3] Browne, E. N., 1909. "The Production of New Hydranths in Hydra by the Insertion of Small Grafts," *J. Exper. Zoöl.*, **7**, 1–24, 66 figs.

[4] Peebles, Florence, 1897. "Experimental Studies on Hydra," *Arch. Entwickelungsmech.*, **5**, 794–819, 34 figs.

INHIBITION OF REGENERATION IN TWO-HEADED OR TWO-TAILED PLANARIANS

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The experiments to be described in this paper have to do with the question of a form-determining dominance of one part of an organism over another. In terms of C. M. Child's theory of individuality, organization or physiological unity in protoplasm is the expression of a system of metabolic gradients. This gradient system is the chief factor in the determination of growth and differentiation. It involves relations of dominance and subordination between different parts of the organism. In general, according to Child, a region of higher metabolic rate exerts "physiological dominance" over a region of lower rate. Whether to be interpreted in terms of gradients or otherwise, such experiments as those described below clearly demonstrate the existence of localized form dominance.

Fresh water planarians (especially Planaria maculata) readily regenerate tissue at any cut surface, unless the fragment of the animal is extremely small. They exhibit a strongly marked and persistent polarity. Except under extraordinary conditions a head regenerates at any anterior cut surface and a tail at any posterior cut surface. But the following experiments show that regeneration at a cut surface may be controlled or

inhibited by the presence of appropriate specialized structures in sufficiently close proximity to it.

Double-headed or double-tailed forms may be produced by operations consisting in splitting the appropriate region of the animal in the median plane. By employment of such double forms it may be shown that the presence of the one head (or tail) affects the regenerative behavior at the cut surface produced by removal of the other head (or tail). If one head of a planarian having not only the head but the anterior third or fourth of the body duplicated be excised so that a considerable stump of its branch of the body is left projecting beyond the common posterior part of the body, regeneration of a head at the cut surface will usually take place (although sometimes at a retarded rate) restoring completely the original double-headed condition. If, however, the cut be made close to the common body so that only a very short stump or none at all is left, retardation or inhibition of head formation may result. It may be shown that the delay or the failure to regenerate is specifically due to the presence of a head. The effectiveness of the inhibitory action of the head seems to vary in an inverse relation to the distance between the head and the cut surface. Other factors, however, than this one of distance are concerned. The consideration of these factors must be deferred to a paper reporting further experimental data.

Methods.—The animals were all Planaria maculata. The worms to be operated on were narcotized in chloretone (0.2%). All operations were performed on a clean glass slide under a dissecting microscope. The cutting was done with the point of a very small sharp scalpel or a fragment of a safety razor blade placed in a handle. The stroke was a sudden downward one as the animal, even if narcotized, reacts when touched and immediately discharges mucus. The cuts were made lengthwise, passing as nearly as possible through the middle of the head or tail and varying extents of the body. As soon as a cut was made, concentration of pigment toward the cut edge was observed. This appearance was doubtless due, at least, in part to general contraction of tissues at the wound, but there was some indication also of an actual migration of pigment toward the raw surface. The return of the pigment to a normal distribution was slow and often was not accomplished until new tissue had regenerated along the cut surface.

It was found that if the incision involved only a small proportion of the length of the body, the cut edges very quickly grew together. Very long cuts (more than two-thirds the length of the animal) generally caused the animal to break into two pieces. To obtain the desired double forms, it was usually necessary to keep the wound open by making successive incisions at the same place and at frequent intervals, even repeating the operation two or three times within one day. When the separation between

the two lateral portions of a part of the animal has become well established (by healing of the wound without fusion of the adjacent cut edges) new tissue develops rapidly along the healed edges and in due time either portion regenerates the structures requisite for the perfection of its bilateral symmetry, resulting in the complete duplication of the region of the animal affected by the operation. The animals were not fed.

Experiments.—Of the forty-seven experiments which were carried out, twenty failed for one reason or another to yield results of significance. Twenty-seven offered more or less convincing evidence of the form-determining dominance of one part over another.

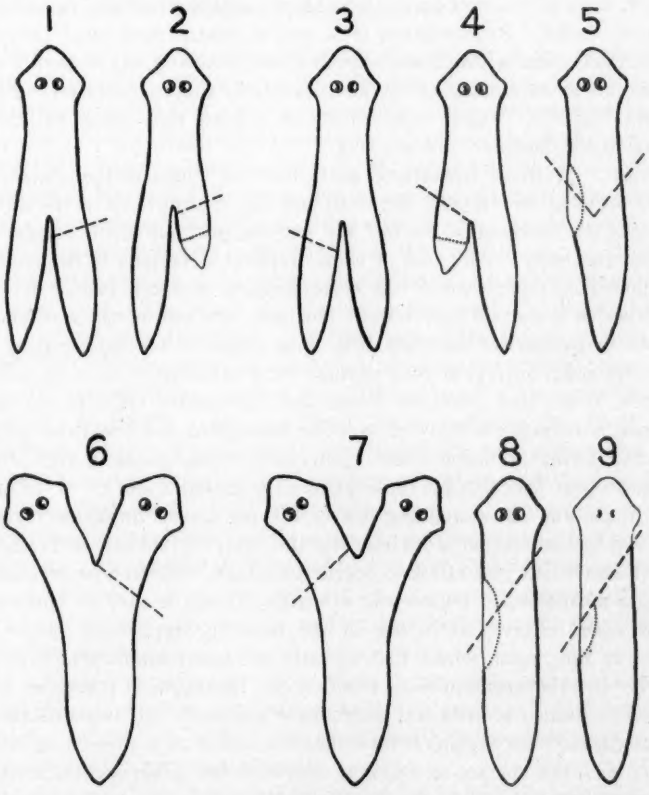
Two types of cases may be recognized. In one (in the following description to be designated as a case of type "A") the results, while entirely consistent with the idea of dominance, yet do not afford absolute proof of it. That is, inhibition of the regeneration of the excised head or tail was not necessarily due to the physiological dominance of the head or tail which remained, but might conceivably have been due to the shock of operation or to lack of food or general exhaustion of regenerative energy, or possibly to other causes. In cases of the other type ("B") appropriate secondary control operations were made in such a way as to prove beyond doubt that the failure to regenerate was really an inhibition due to the dominance of the existing head or tail and not a consequence of any of the other conditions suggested above.

The general nature of the results of the experiments will be indicated by describing one of each of the two types of cases of head inhibition and one of each of the two types of tail inhibition.

I. TAIL INHIBITION, CASE A.—A double-tailed animal was produced. The right tail was excised leaving a short stump (Fig. 1). The stump regenerated a new tail. Sixteen days after the first amputation, the regenerated tail, along with the stump bearing it, was excised close to the common anterior part of the body (Fig. 2). No tail was regenerated. A little new tissue developed at the wound region tending to restore the outline of the body of a normal individual.

CASE B. From a double-tailed animal, the left tail was excised leaving a short stump (Fig. 3). Regeneration of a tail began. The regenerating tail, together with the original stump, was then excised (Fig. 4). Seven days later there were no indications of regeneration. New tissue had formed at the wound region and the body had acquired a fairly normal outline. The animal was then divided into two parts by a cut shaped like the letter "V" (Fig. 5), the left arm of the "V" passing through the new tissue resulting from the last operation and the right arm of the "V" severing what had been the right portion of the originally bifurcated body. Three days later regeneration was actively in progress at the cut surface of both parts of the worm. Eventually the posterior part developed a

single normal head and the anterior part developed a single symmetrically formed tail. The tissue intersected by the left arm of the V-shaped cut was tissue where, following the earlier operation, regeneration had failed. Yet, after this later operation, regeneration of the tail proceeded at equal rate on the right and left sides.



FIGURES 1–9.

Regenerated regions are indicated by lighter lines. Dotted lines indicate limits between old and regenerated tissue. Broken lines indicate planes of tran-section.

II. HEAD INHIBITION, CASE A.—From a double-headed animal, the right head-bearing portion was excised, leaving no projecting stump (Fig. 6). A little new tissue developed at the wound region. No head regenerated. The outline of the right side of the animal gradually straightened out and, sixteen days after the amputation of the right head, the animal had the form of a normal individual.

CASE B.—From a double-headed animal, the left head was excised leaving only a very slight projecting stump (Fig. 7). No head was regenerated and, thirty-one days after the amputation, the animal appeared like a normal individual (Fig. 8). Two days later the remaining (original right) head was excised by a cut extending obliquely across the animal (Fig. 8) so as to intersect on the left side the region where regeneration had formerly failed. Regeneration of a single symmetrical head proceeded without delay and within a week the new head was well established (Fig. 9). On the ninth day following the last operation, this regenerated head was excised (Fig. 9). Again regeneration of a head took place without retardation or abnormality of any sort.

Results.—Splitting operations performed on Planaria maculata result in the duplication either of the head and varying portions of the anterior region of the body, or of the tail and varying portions of the posterior region of the body. Any one of these duplicated regions is the complete morphological equivalent of the corresponding antero-posterior extent of the body of a normal individual. Further, any one of these duplicated regions possesses, within itself, the same capacity for regeneration that the corresponding region of a normal body possesses. If, however, one of these duplicated parts be transected sufficiently close to the single common portion of the body, then (other conditions, not now to be specified in detail, being favorable) the regenerative replacement of that excised duplicate part may exhibit retardation or complete failure. If, following such inhibition, the remaining one of the previously duplicate regions is removed by transection at its base and the cut is carried across through the region which had just failed to regenerate, there will then be regenerated a single normal body region and extremity (head or tail) of appropriate polarity and equivalent to one of the formerly duplicated parts. The tissues in the region which had recently exhibited inhibition participate actively in this regeneration. Further, the literature of planarian regeneration contains experimental data ample to justify the assertion that inherent capacity for regeneration of head or tail ends is present in relation to *any* such cut surface as those at which, in our experiments, inhibition has been shown.

It is, therefore, proved that the inhibition is not due to shock of operation, pathological condition of tissues or general exhaustion of regenerative capacity, but is specifically due to the presence, in appropriate proximity and relations, of the *equivalent* of the structures which had been removed. It is thus shown that the regenerative behavior at a cut surface of a planarian is influenced by the presence or absence of structures at a region more or less remote from the wound. There is demonstrated a "dominance"—but an inhibitory dominance—of one part of an organism over another.

INHIBITION OF REGENERATION IN PLANARIANS BY GRAFTING: TECHNIQUE OF GRAFTING

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In Hydra (of various species), following transection at any level of the column, a complete head will be developed at the exposed distal (oral) cut surface, provided that the fraction of the column represented by the piece is above a certain minimum at which no regeneration of any sort may be expected. Experimentation upon Hydra carried on at the Zoölogical Laboratory of Harvard University[1] has demonstrated that regeneration of a head at an oral cut surface of a hydra column may be inhibited by the presence of a head grafted onto the side of the column. E. N. Browne[2] grafted a small portion of hydra peristome tissue into the column of another individual. The result was the development of a hydranth at the point of insertion of the graft. The material of the new hydranth was derived from the column which bore it except that the grafted tissue, not perceptibly increased in quantity, constituted a part of the peristome region of the hydranth. Only peristome tissue produces this result. It is, therefore, suggested that agencies localized in the head region of a Hydra may stimulate column tissue to produce a head or, under different circumstances, may inhibit head regeneration.

Planaria maculata, like Hydra, possesses extraordinary powers of regeneration. Transection at any level of the body, whether in an exactly transverse plane or an oblique one, will be followed by regeneration of a head at the exposed anterior cut surface provided that the fragment of body be not too small.

The following experiments upon Planaria maculata were undertaken with a view to finding out whether, as in Hydra, regeneration of a head at an exposed anterior cut surface may be inhibited by the presence of a head grafted relatively near the cut surface.

Grafting Technique.—Little grafting work on planarians has been done owing doubtless to the difficulties of the technique. T. H. Morgan[3] reports two successful grafting operations on the large planarian, Bipalium. L. V. Morgan[4] describes many successful operations on Phagocata gracilis and ten cases on Planaria maculata. The latter author developed a technique whose essential feature was that the pieces to be united were confined in proper contact with one another between slips of damp tissue paper.

The difficulties of the grafting operation consist in the extreme softness and delicacy of the tissue, its propensity for disintegration (especially if

an anesthetic is used), the mobility of fragments and the contamination of the cut surface with mucus. It is essential that freshly cut surfaces shall be brought immediately into close contact with each other and held firmly in that relation for several hours.

For our purpose of grafting so small a piece as a head onto the side of the body of another worm, the tissue paper method, upon trial, gave no promise of success. (The experiments were confined to Planaria maculata which was the only species abundantly available at the time.) Another method was devised and proved so far successful as to justify the following account of it.

The grafting to be described in this paper was done by use of gelatin for confinement of the pieces which were to be united. The solution used was 3.5% of pure acid-free gelatin (Eimer and Amend, New York) in distilled water. This solution was allowed to cool in the dish in which it had been heated until it began to stiffen. It was then poured out onto cold glass slides where it set forming a layer about a millimeter thick over the glass. For a grafting operation two planarians, preferably narcotized (0.2% chloretone), were taken. From the side of the body of one was excised a piece of tissue leaving a wound appropriate in size for reception of the graft. The wound was made deep enough to interrupt the lateral nerve cord. This animal having been placed on the prepared gelatin surface, immediately the head was excised from the other animal and placed with its cut surface against the wound surface of the first animal. Then strips and fragments of appropriate shape and size, cut from the fairly stiff but moist gelatin, were packed tightly around and over the two members of the operation with a view to preventing their separation. The slide was then placed in a moist chamber and left there for eighteen to twenty-four hours, after which the gelatin was very carefully picked away and the graft compound was transferred to a small dish of its natural pond water.

An alternative method which promised some success consisted in literally imbedding the graft members in gelatin which, while still fluid, was fairly cool and just at the point of setting. It is difficult, however, to hit precisely the right moment. In some cases, too, presumably some mucus was thrown off which, diluting the neighboring gelatin, resulted in the formation of a fluid vacuole in the midst of the otherwise stiff gelatin. In other cases failure was apparently due to the gelatin becoming too stiff.

Solutions of agar were tried in place of gelatin. Agar has the advantage over gelatin of setting more quickly—although sometimes too quickly—but is inferior in transparency and seemed to result in greater mortality. Gelatin, on the whole, was preferable.

Incidentally some observations were made upon the viability of the planarians packed in gelatin or imbedded in it. In the latter case the thin-

nest possible layer of gelatin intervened between the animal and the air. One animal, confined in cooled 3.5% gelatin, remained alive six days. In several cases animals lived four or five days in gelatin of between 2.5% and 3.66%. One animal lived four days in 2.5% agar, and one lived five days on a floor of gelatin and covered with agar. At the ends of the periods just mentioned the animals were liberated from the gelatin or agar.

The very brief time available for this work was mostly spent in developing the technique. Eight successful grafts of a head onto the side of a body were obtained.

Inhibition of Regeneration by Grafting.—The general plan of the experiment consisted in grafting the head of one animal onto the side of the anterior portion of the body of another. After the graft had become well healed in place, the stock animal's original head was removed. This amputation leaves upon the body an exposed anterior cut surface at which, if no operation other than the beheading had been performed, regeneration of a head would surely occur. Will the presence of the foreign head in close proximity to the wound affect the regeneration?

CASE I.—A head piece whose length was about twice its width was grafted into the side of the body of another animal a little in front of the pharynx region. The grafted head was smaller than the stock head. Five days after the operation the original head was excised a little posterior to the eyes (Fig. 1) and leaving a considerable stump of body projecting beyond the place of insertion of the graft. A head regenerated (Fig. 2), and on the twelfth day after the grafting operation this head was excised by a transverse cut, leaving a short stump projecting beyond the graft (Fig. 2, *a*). Again regeneration of a head occurred, and on the thirty-fourth day the then fully developed head was excised by an oblique cut leaving no stump beyond the graft (Fig. 2, *b*). At this oblique cut surface a head developed. On the fifty-second day (Fig. 3) this third regenerated head was excised at the level indicated in the figure. Again a head regenerated. On the seventy-first day (Fig. 4) the new head was removed at the level indicated in the figure. Again regeneration occurred. This fifth regenerated head in the course of two weeks became well established. No further operation was performed. But on the twenty-ninth day after the last beheading (the one-hundredth day of the experiment) it was noted that the portion of the graft head anterior to the eyes was undergoing disintegration. Within the next three days the graft head suffered complete histolysis and meanwhile the regenerated head swung around so that its axis came into line with the axis of the body. The general appearance was that of a planarian of normal form.

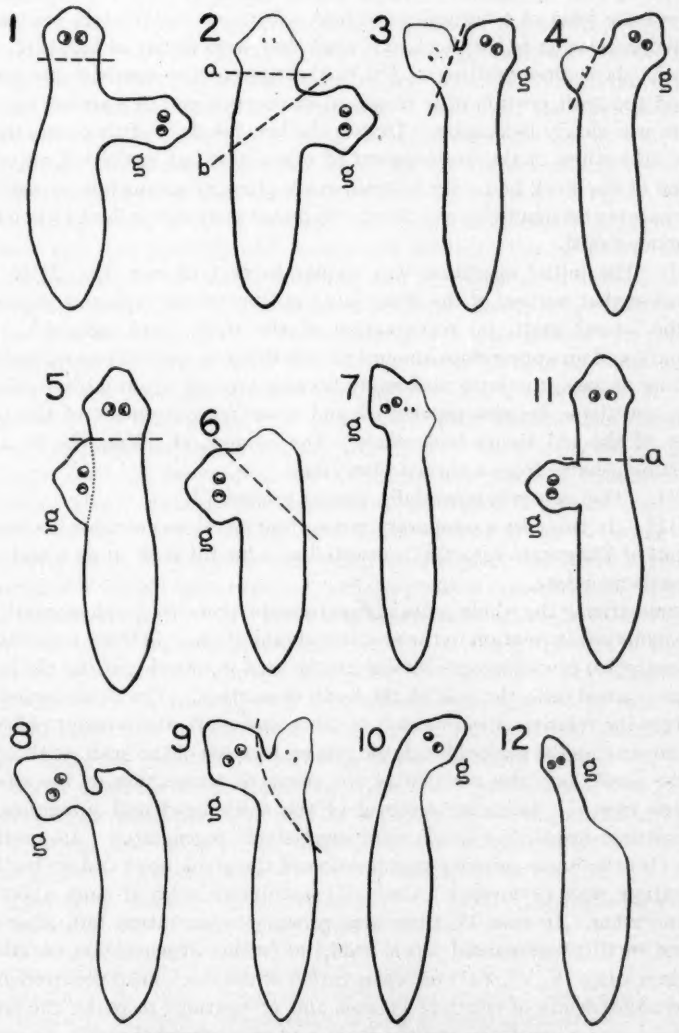
II. This case was essentially similar to case I as far as the primary regeneration of the stock head, but no further operation was performed.

III. A head, excised just behind the eyes, was grafted into the side

of the anterior part of the body of another animal, as shown in figure 5. On the third day after the operation the original head was excised at a transverse plane just in front of the graft. During the next seventeen days a small outgrowth of new tissue at the cut surface assumed the form of an abortive head. Certain small pigment spots were doubtfully recognized as eyes (Fig. 6; doubtful eye spots not shown). On the twentieth day after the initial operation the new part was removed together with some old tissue, the plane of cutting being oblique to the body axis (Fig. 6). During the next two weeks considerable new tissue developed at the cut surface and at times it exhibited a form suggestive of a head. Early in the fourth week (the seventh since the initial operation) two pigment spots (Fig. 7) were suspected of being eyes, but a week later one of them (the left) had disappeared. The graft head had swung around so that its axis was nearly in line with the body axis. In the eighth week of the experiment the animal died without having exhibited further change.

IV. Two worms of very unequal size were taken. The head end of the larger, excised so that its length was about equal to the width of the head, was grafted into the left side of the body of the smaller and at such a position that the tips of the two heads were about equidistant from the region of junction. Probably because of imperfect initial union, the connection between graft and stock at first appeared as a narrow neck-like region. After the second week following the operation this connecting region began slowly to increase in width. On the fourteenth day after the grafting operation the stock head was excised. Figure 8 shows the condition soon after the beheading. A very much retarded regeneration of the stock head ensued, eight days elapsing before definite evidence of head regeneration could be detected. In the course of two weeks more the head became fairly well regenerated although the eyes were still very small. On the fortieth day after the grafting operation the regenerated stock head was removed as indicated in figure 9. In the course of three weeks following this last operation, a considerable amount of new tissue developed at the region of the cut surface, but it gave rise to no projection indicative of a head. Figure 10 shows the appearance of the animal on the fourteenth day after the last operation. With lapse of more time, the concavity in the lateral outline at the region of the wound was gradually filled in and the axis of the graft head came more and more nearly into alignment with the stock axis until, eventually, the form of the graft compound was hardly distinguishable from that of a normal worm. Without exhibiting further change of form, the animal lived to the thirtieth day after the last operation.

V. A grafting operation produced the compound represented in figure 11. Four days after the operation the stock was transected at *a*. Aside from the formation of a narrow band of new tissue at the cut surface,



FIGURES 1–4, CASE I. FIGURES 5–7, CASE III.
FIGURES 8–10, CASE IV. FIGURES 11–12, CASE V.

a, *b*, planes of cutting; *g*, grafted head. Regenerated regions indicated by lighter lines.

no regeneration occurred. The compound lived sixty-three days after the beheading of the stock. During this time there was very slow regulation toward the form of a normal individual. The axis of the graft gradually shifted in relation to the stock axis until they were nearly in line (Fig. 12; fifty-six days after grafting). Up to the time of the death of the compound the graft portion of it remained disproportionately narrow, but its width was slowly increasing. During the last ten days of this case, there were indications of the development of a new pharynx in the left anterior region of the stock body, the original stock pharynx meanwhile persisting. The nature and significance of this development were still in doubt when the compound died.

VI. The initial operation was similar to that of case V. After removal of that portion of the stock lying in front of the region of junction of the lateral graft, no regeneration of the stock head occurred. By formation of an appropriate amount of new tissue at the cut region, and by shifting of axes, the form ultimately became that of a normal individual. The new tissue became pigmented and some rearrangement of the pigment of the old tissue took place. The compound ultimately became indistinguishable from a normal planarian.

VII. This case was essentially similar to case VI.

VIII. In this case a temporary union (four days) was obtained between a head of Phagocata (gracilis?), inserted as a lateral graft upon a body of Planaria maculata.

Summarizing the whole series of experiments, there are seven cases which are significant in relation to the problem of inhibition. In these seven cases the operative procedure was similar insofar as it involved grafting the head of one animal onto the side of the body of another. The cases varied as regards the relative sizes of heads of stock and graft, the amount of body accompanying the grafted head, the precise location of the graft on the side of the stock, and the position of the plane of transection of the stock. In one case (I), following removal of the stock head and subsequently regenerated heads, five heads were successively regenerated. In another case (II) there was primary regeneration of the stock head and no further operation was performed. Case III exhibited delayed and abortive regeneration. In case IV there was primary regeneration but, after removal of this regenerated stock head, no further regeneration occurred. In three cases (V, VI, VII) no regeneration of the stock head occurred and, by readjustments of relations of axes and proportions of parts, the compound, consisting of the grafted head and the beheaded stock, assumed more or less exactly the appearance of a normal individual. In two of the three cases the compound ultimately became indistinguishable from a normal individual.

Why is regeneration sometimes inhibited? Case I in which the worm,

having endured the grafting operation, regenerated five successive times, is enough to justify rejection of the idea that inhibition may be due to mutilation or shock of operation or exhaustion of regenerative capacity. In the cases where inhibition occurred, the stock exhibited not the least indication of pathological conditions of any sort. In all of the experiments the anterior cut surfaces produced by beheading the stock, whether such cut surfaces were transverse or oblique, were surfaces at which regeneration was to have been expected. In the absence of any operative complication beyond the mere beheading, regeneration certainly would have taken place in every instance unless, as occasionally happens, infection or other agencies had caused the death of the beheaded worm. It is, therefore, a reasonable inference that the retardation and inhibition of regeneration exhibited in certain of the experiments is specifically due to the presence of the graft. The explanation of the variability as regards regeneration and inhibition is to be found, we believe, in the varying relations of the graft and stock. Compare the initial conditions in cases I and V. In I (Fig. 1) the graft head was slightly smaller than the stock head, a considerable length of body accompanied the graft head, the graft was relatively far forward on the stock, and the plane of transection of the stock was at some distance anterior to the region of attachment of the graft. In case V (Fig. 11) the two heads were of about the same size, relatively little of the body accompanied the graft head, the region of grafting was farther back than in I, and the plane of transection of the stock was close to the graft. In I there was regeneration and in V there was inhibition. (It is perhaps significant, too, that in the latter part of the experiment in case I the grafted head suffered a histolytic degeneration of some sort.) Experimental data which cannot be reported in this paper indicate that all of these relations of graft and stock may be concerned in determining the regenerative behavior of the beheaded stock. The point of special significance established by these few experiments is that regeneration of the planarian head may be inhibited by a foreign head brought, by grafting, into appropriate relation to the wound region of the beheaded worm. Further data will be reported in other papers.

[1] Rand, H. W., Bovard, J. F., and Minnich, D. E. 1926. "Localization of Formative Agencies in Hydra," these PROCEEDINGS, **12**, 565–570.

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[4] Morgan, L. V. 1906. "Regeneration of Grafted Pieces of Planarians," *J. Exper. Zoöl.*, **3**, 269–294, 17 figs.